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A Certified Numerical Approach to Describe the Topology of Projected Curves

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Résumé

Consider a plane curve $\mathcal{B}$ defined as the projection of the intersection of two algebraic surfaces in $\mathbb{R}^3$. In general, $\mathcal{B}$ has singular points and thus is not a manifold. We aim at computing a topologically exact representation of $\mathcal{B}$ within a box of the plane using certified numerical algorithms. In a recent work, authors show how to describe the set of singularities Σ of $\mathcal{B}$ as regular solutions of a so-called ball system suitable for a numerical subdivision solver. Here, the space curve is first enclosed in a set of boxes with a certified path-tracker to both restrict the domain where the ball system is solved and determine $\mathcal{B} \setminus \Sigma$. A combinatorial representation of $\mathcal{B}$ is finally given, encoded as a non-embedded graph. We experimented our method and got promising results.

1. Introduction

Let $\mathcal{C}$ be a smooth space curve defined as the intersection of two algebraic surfaces $P = Q = 0$, with P, Q polynomials in $\mathbb{R}[x,y,z]$. We aim at computing the geometry and the topology in a compact domain $\mathbf{B}_0 \subset \mathbb{R}^2$ of $\mathcal{B} = \pi_{(x,y)}(\mathcal{C})$ where $\pi_{(x,y)} : \mathbb{R}^3 \to \mathbb{R}^2$ denotes the projection in the (x,y)-plane. In general, $\mathcal{B}$ is not smooth and has singular points, *i.e.* points where $\mathcal{B}$ has no well defined tangent direction. Generically, the only singular points of a projected curve are transversal crossings of two branches of the curve, called nodes.

By computing the geometry of $\mathcal{B}$, we mean being able to draw it with an arbitrary precision. We address this problem by tracking, with a certified interval path tracker, the space curve $\mathcal{C}$. The output of this step is a sequence of boxes (*i.e.* multi-dimensional extensions of intervals) of width as small as desired which union encloses the space curve, and can be projected in the plane to draw $\mathcal{B}$.

By computing the topology we mean first isolating the singular points of $\mathcal{B}$, that is giving a set of boxes of $\mathbf{B}_0$ such that each singularity of $\mathcal{B}$ in $\mathbf{B}_0$ lies in a box, and each box contains a unique singularity. In a previous work, we have shown how the singularities can be described as the regular solutions of a so-called ball system involving 4 equations in 4 unknowns. This characterization does not use resultant or sub-resultant theory, which main disadvantage is involve large polynomials in term of degree, bit-size and number of monomials. The ball system can be solved within $\mathbb{R}^4$ with a certified numerical solver based for instance on intervals and subdivision. We use here the geometry of the space curve, that is the boxes enclosing it, to restrict the solving domain of the ball system. Experiments show the efficiency of this approach to isolate singularities of $\mathcal{B}$.

Then, we show how encoding the topology of $\mathcal{B}$ in a combinatorial structure to describe its homeomorphism class. It is done by locating uniquely the nodes of the projections in two boxes enclosing the space curve. The structure encoding the topology proposed here is a non-embedded graph.

A special occurrence of our problem is the case where $Q = P_z$, where P_z denotes the derivative of P with respect to z. The projected curve $\mathcal{B}$ of $\mathcal{C}$ defined by $P = P_z = 0$ is called the apparent contour of the surface $P = 0$. Generic singular points of $\mathcal{B}$ are nodes and cusps, *i.e.* projections of points where $\mathcal{C}$ has a vertical tangent. Fig. 1 shows, for a torus $P = 0$, its intersection with the surface $P_z = 0$ in bold line and its apparent contour that has cusp and node singularities.

The paper is organized as follows. Sec. 2 describes how the curve $\mathcal{C}$ is enclosed. Sec. 3 defines the ball system and shows how the enclosure of $\mathcal{C}$ is used to restrict its solving domain. It also present an algorithm to determine the type of a singularity (node or cusp) isolated with the ball system, when $\mathcal{B}$ is an apparent contour. Sec. 4 is dedicated to the computation of the graph encoding the topology of $\mathcal{B}$. Sec. 5 reports experiments on the isolation of the singularities of $\mathcal{B}$ with our approach, and Sec. 6 concludes. The remaining of this section presents previous and related works, defines formally objects we aim at computing, basics about interval arithmetic and certified numerical methods.

1.1. Previous works

State-of-the-art symbolic methods that compute topology of plane real curves defined by polynomials mainly use resultant and sub-resultant theory to isolate critical points [Hon96, MPS*06]. There are some alternatives, using for instance Groebner bases and rational univariate represen-

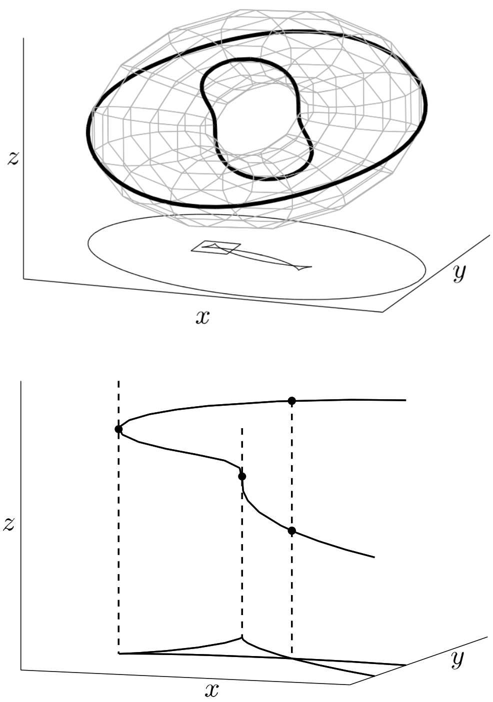

Figure 1: *Top: a torus $P = 0$, in bold line the curve $P = P_z = 0$, its apparent contour, and a zoom zone. Bottom: a detail, with antecedents by projection of cusps and nodes singularities.*

tations [CLP*10]. Once singularities are isolated, the local topology around singularities is computed, and they are connected together with a sweeping algorithm. The topology is thus encoded as an embedded graph that describes the ambient homeomorphism class of the curve.

Numerical methods together with interval arithmetic are able to compute and certify the topology of a non-singular curve when the interest area is a compact subset of the plane [MGGJ13, KX94, PV04]. However they fail near any singular point of the curve. Isolating singularities of a planar curve $f(x,y) = 0$ with a numerical method is a challenge since the set of singular points is described by the non-square system $f = \frac{\partial f}{\partial_x} = \frac{\partial f}{\partial_y} = 0$, and singularities are not necessarily regular solutions of this system. The latter system can be translated into a square system using combinations of its equations with first derivatives [Ded06], and non-regular solutions can be handled trough deflation systems [OWM83, LVZ06], but the resulting systems are usually still overdetermined.

Starting with the work of Whitney [Whi55], the catastrophe theory was developed to classify the singularities arising while deforming generic mappings (see [AVGZ88, Dem00] for example). From an algorithmic point of view, the authors of [DL13] use elements of the catastrophe theory to derive an algorithm isolating the singularities arising in mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$.

More specifically, the problem of isolating the singularities of the projection of a generic algebraic space curve was investigated in [IMP15b]. The authors use resultant and subresultant theory to represent the set Σ of singularities as the solutions of a regular bivariate system suited to a branch and bound solving approach. To overcome the drawbacks of resultant and sub-resultant, [IMP15a] studies the geometric configurations of the space curve that induce singularities on the projected curve, and describes elements of Σ as the regular solutions of the ball system involving 4 equations in 4 unknowns.

1.2. Notations and definitions

Lowercase boldface letters denote real intervals and uppercase boldface letters boxes, that are vectors of intervals. Let $\mathbf{x}$ be a real interval, $l(\mathbf{x})$ denotes its lower bound, $u(\mathbf{x})$ its upper bound and $w(\mathbf{x})$ its width defined as $u(\mathbf{x}) - l(\mathbf{x})$. Let $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ be a box, $\partial\mathbf{X}$ denotes the boundary of $\mathbf{X}$ that is the set $(l(\mathbf{x}_1), \ldots, \mathbf{x}_n) \cup (u(\mathbf{x}_1), \ldots, \mathbf{x}_n) \cup \ldots \cup (\mathbf{x}_1, \ldots, l(\mathbf{x}_n)) \cup (\mathbf{x}_1, \ldots, u(\mathbf{x}_n))$. $w(\mathbf{X})$ is defined as $max_{1 \leq i \leq n} w(\mathbf{x}_i)$.

For a polynomial P in $\mathbb{R}[x_1, \ldots, x_n]$, P_{x_i} denotes its partial derivative with respect to x_i, and $P_{x_i x_j}$ its derivative with respect to x_i and x_j. Let $P_1, \ldots, P_n \in \mathbb{R}[x_1, \ldots, x_n]$, a solution x of $P_1 = \ldots = P_n = 0$ is said to be *regular* if the jacobian matrix $A = [a_{i,j}]$ where $a_{i,j} = (P_i)_{x_j}$ evaluated in x has full rank. Otherwise x is said to be *singular*.

Cursive letters will denote sets of points. In this paper we mostly work with points, boxes and curves in $\mathbb{R}^2$, $\mathbb{R}^3$ or $\mathbb{R}^4$. We use the following naming scheme: objects in $\mathbb{R}^2$ are named with the letter B, in $\mathbb{R}^3$ with the letter C and in$\mathbb{R}^4$ with the letter D.

1.3. Objectives

Consider now 2 polynomials P, Q in 3 unknowns, and the set $\mathcal{C} = \{p \in \mathbb{R}^3 | P(p) = Q(p) = 0\}$. Our goal is to compute both a geometrical and a topological description of the projection $\mathcal{B} = \pi_{(x,y)}(\mathcal{C})$ in a box $\mathbf{B}_0 = (\mathbf{x}_0, \mathbf{y}_0)$ of $\mathbb{R}^2$. Let $\mathbf{C}_0$ be the set of $\mathbb{R}^3$ above $\mathbf{B}_0$ (*i.e.* $\mathbf{C}_0 = \mathbf{B}_0 \times \mathbb{R}$), and $\partial\mathbf{C}_0$ be the boundary of $\mathbf{C}_0$, that is the set $\partial\mathbf{B}_0 \times \mathbb{R}$. In what follows, $\mathcal{C}$ will denote $\mathcal{C} \cap \mathbf{C}_0$ and $\mathcal{B}$ will denote $\mathcal{B} \cap \mathbf{B}_0$.

We assume that $\mathcal{C}$ satisfies the assumptions in Subsec. 2.1 and 3.1. These assumptions are generic in the sense that they are met for almost all choices of curves and coordinate systems, see for instance [Dem00] for the concept of genericity. According to our assumptions, $\mathcal{C}$ intersects the boundary $\partial\mathbf{C}_0$ in a finite set of points. Hence $\mathcal{C}$ has several connected components $\mathcal{C}^1, \ldots, \mathcal{C}^n$ each being either diffeomorphic to a circle, or to the segment $[0,1]$.

Definition 1 (ε-approximation) We say that a sequence of boxes $(\mathbf{C}_i)_{i=1}^m$ is an ε-approximation of a connected component $\mathcal{C}^k$ if $\mathcal{C}^k \subset \bigcup_{i=1}^m \mathbf{C}_i$, and for all $1 \leq i \leq m$:

(i) $\mathcal{C}^k \cap \mathbf{C}_i$ is diffeomorphic to $[0,1]$,
(ii) $w(\mathbf{C}_i) \leq \varepsilon$,
(iii) $\mathcal{C}^k \cap (\mathbf{C}_i \cup \mathbf{C}_{i+1})$ is diffeomorphic to $[0,1]$ for $i < m$,
(iv) $\forall j < i-1$, $\mathbf{C}_i \cap \mathbf{C}_j = \emptyset$ for $i < m$

Moreover, if $\mathcal{C}^k$ is diffeomorphic to a loop, we require that $\mathcal{C}^k \cap (\mathbf{C}_m \cup \mathbf{C}_1)$ is diffeomorphic to $[0,1]$, otherwise, $\mathbf{C}_m \cap \mathbf{C}_1 = \emptyset$.

This definition allows us to draw an algebraic curve with the guarantee that any point of the drawing is at most at distance ε from the original curve, and reciprocally. Obviously the projection of such a drawing of the curve $\mathcal{C}$ is also a good geometric approximation of the projected curve. Top part of Fig. 2 represents ε-approximations of the two connected components of the curve $\mathcal{C}$ over $\mathbf{B}_0$ for the example of the torus and their projections.

If we are interested in preserving topological properties, then we need to compute a graph homeomorphic to the curve.

A labeled graph G is a triplet of 3 finite sets (V,E,L) equipped with 2 functions $e : E \to V \times V$ and $\gamma : V \to L$. The set $V = \{v_1,\dots,v_n\}$ is the set of vertices of G, $E = \{e_1,\dots,e_m\}$ is the set of edges of G and L are the labels of the vertices of G. The application e defines the 2 extremities of each edge, and γ defines the labels of each vertex. Note that with this definition, edges of G can be loops (*i.e.* $e(e_i) = (v_j, v_j)$) and several edges can connect two vertices (*i.e.* $e(e_i) = e(e_j)$). If we identify each edge with the segment $[0,1]$, a graph can be naturally seen as a topological space ([GT87, §1.3.2, p. 18]).

Moreover under our assumptions, the singularities of the projected curve are either nodes or cusp and we want to recover this information on the graph with labels.

Definition 2 (Graph homeomorphic to the projection) We say that a labeled graph G is homeomorphic to the projection of $\mathcal{C}$ if the topological space associated to G is homeomorphic to $\pi_{(x,y)}(\mathcal{C})$, and each vertex is labeled *cusp* (resp. *regular* or *node*) if the corresponding point in the algebraic curve is a *cusp* (resp. *regular* or *node*) point.

Bottom part of Fig. 2 represents such a graph G homeomorphic to the apparent contour of a torus. In the next sections, we will show how to compute efficiently these 2 kinds of outputs.

1.4. Numeric certified tools

Numeric certified tools used here are based on interval arithmetic (see [Neu90, Kea96, MKC09, Sta95]), that is a way of computing with intervals (which endpoints are floating numbers) instead of computing with floating numbers, while carefully handling rounding to overcome numerical approximations that naturally occur with floating number arithmetic.

1.4.1. Interval arithmetic

Usual arithmetic operations such as additions, multiplications and so on can be extended to intervals and boxes. Letting $*$ being an operator and $\circledast$ its interval extension, one has $\mathbf{X} \circledast \mathbf{Y} \supseteq \{x * y | x \in \mathbf{X}, y \in \mathbf{Y}\}$.

Polynomials, seen as combinations of such operations, can thus be evaluated over intervals or boxes. If P_1 is a polynomial and $\mathbf{X}$ a box, the evaluation of P_1 over $\mathbf{X}$ results in an interval $\mathbf{Y}$ that satisfies $\mathbf{Y} \supseteq \{P_1(x) | x \in \mathbf{X}\}$, and in general

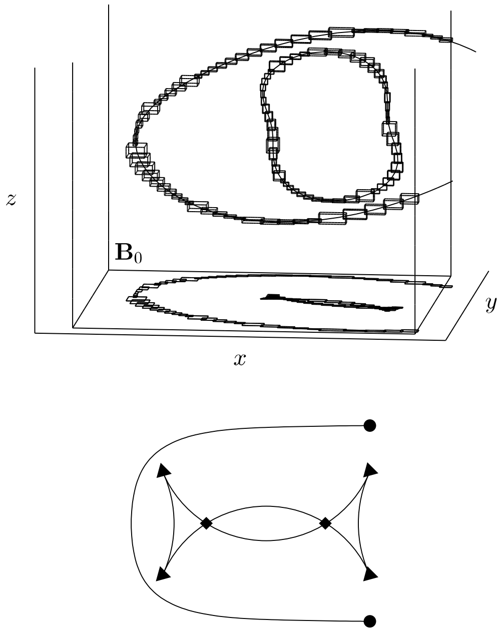

Figure 2: *Top: ε-approximations of components of $\mathcal{C}$ over a box $\mathbf{B}_0$, and their projections. Bottom: A graph G (more precisely its associated topological space) homeomorphic to the projection of $\mathcal{C}$. Black circles represent vertices of G labeled regular, black diamonds represent vertices of G labeled node and black triangles represent vertices of G labeled cusp.*

one has $\mathbf{Y} \supset \{P_1(x) | x \in \mathbf{X}\}$. As a consequence, if $0 \notin \mathbf{Y}$ then one can certify that P does not have any root in $\mathbf{X}$.

1.4.2. Criteria for existence and uniqueness of solutions

Let $P_1,\dots,P_n$ be n polynomials in n unknowns, $S = \{P_1 = 0,\dots,P_n = 0\}$ the associated system of equations. A box $\mathbf{X} \subset \mathbb{R}^n$ *isolates* a solution of S if there exists a unique $x \in \mathbf{X}$ such that $P_1(x) = \ldots = P_n(x) = 0$.

Several criteria can be found in the interval arithmetic literature that certify existence and uniqueness of a solution of S in a box, see for instance [Neu90, Kea96, MKC09, Sta95]. Most of them are based on the Brouwer fixed point theorem and use interval Newton operators that contract a box around a solution.

Letting F be the multi-variate function with components $P_1,\dots,P_n$ and $\mathbf{X}$ an interval of $\mathbb{R}^n$, interval Newton operators are of the form $N(\mathbf{X}) = y + \mathbf{V}$, where $y \in \mathbf{X}$, and $\mathbf{V}$ is a box containing solutions of the linear system $J(\mathbf{X})v = F(y)$ where $J(\mathbf{X})$ is the interval evaluation of the jacobian matrix of F. Among other interval Newton operators is the Krawczyk operator ([Kra69, Kea96]) that takes y as the middle of $\mathbf{X}$ and an approximate inverse of the derivative of F in y to determine the box $\mathbf{V}$. Let us note K_S the Krawczyk operator for the system S. Important results about Krawczyk operator used in interval numerical tools are:

Algorithm 1 `IsolateSols(`S`,`$\mathbf{X}_0$`)`

Input: An initial domain $\mathbf{X}_0 \subseteq \mathbb{R}^n$ possibly unbounded, a system S of n polynomials in n unknowns with only regular solutions in $\mathbf{X}_0$, a contractor K_S.

Output: A set X^* of boxes such that:

- boxes of X^* are pairwise disjoint,
- x is a solution of $S \Rightarrow \exists \mathbf{X} \in \mathrm{X}^*$ s.t. $x \in \mathbf{X}$,
- $\mathbf{X} \in \mathrm{X}^* \Rightarrow \mathbf{X}$ contains a unique solution of S
- $\mathbf{X} \in \mathrm{X}^* \Rightarrow K_S(\mathbf{X}) \subset int(\mathbf{X})$.

- $K_S(\mathbf{X}) \subset int(\mathbf{X}) \Rightarrow \mathbf{X}$ contains a unique solution to S.
- $K_S(\mathbf{X}) \subset int(\mathbf{X}) \Rightarrow$ the sequence $\mathbf{X}^{(0)} = \mathbf{X}$, $\mathbf{X}^{(i+1)} = K_S(\mathbf{X}^{(i)})$ converges quadratically.
- $K_S(\mathbf{X}) \cap \mathbf{X} = \emptyset \Rightarrow$ there is no solution to S in $\mathbf{X}$.

where $int(\mathbf{X})$ is the interior of $\mathbf{X}$, *i.e.* the set $\mathbf{X} \setminus \partial\mathbf{X}$

In what follows we will call *contractor* an operator that verifies these three properties.

1.4.3. Certified numerical isolation

Interval evaluation of polynomials, Krawczyk operator and bisection of boxes can be used together to design a simple certified numerical method to isolate all solutions of S in a given initial box $\mathbf{X}_0$. Such methods are described for instance in [Neu90, Kea96, MKC09, Sta95] and called interval branch and bound algorithms or subdivision methods. Making abstraction of the numerical precision used in implementation, they terminate with correct results when all solutions of S in $\mathbf{X}_0$ are regular. Note that it is possible to extend such branch and bound methods to unbounded initial domains, see [Neu90, Sec. 5.6] or [Sta95][Sec. 5.10].

In what follows, we will consider the procedure `IsolateSols(.,.)` with specifications given in Algorithm 1, that could be implemented as described above.

2. Enclosing $\mathcal{C}$

In this section we show how to enclose each connected component of $\mathcal{C}$ using initial boxes and how to combine the different connected components.

2.1. Assumptions

Definition 3 A point $p \in \mathcal{C}$ is x-critical if the x component of the tangent of $\mathcal{C}$ at p vanishes. x-critical points of $\mathcal{C}$ are the solutions of the system $P = Q = R = 0$, where $R = P_yQ_z - P_zQ_y$.

Recall that $\mathbf{B}_0 = (\mathbf{x}_0, \mathbf{y}_0)$ and $\mathbf{C}_0 = \mathbf{B}_0 \times \mathbb{R}$. We make the following assumptions on P, Q and $\mathcal{C}$:

(A_1) The curve $\mathcal{C}$ is smooth above the box $\mathbf{B}_0$.
(A_2) $\mathcal{C}$ is compact over $\mathbf{B}_0$.
(A_3) $P(x_*, y, z) = Q(x_*, y, z) = 0$ has finitely many regular solutions when $x_* = l(\mathbf{x}_0)$ or $u(\mathbf{x}_0)$.
$P(x, y_*, z) = Q(x, y_*, z) = 0$ has finitely many regular solutions when $y_* = l(\mathbf{y}_0)$ or $u(\mathbf{y}_0)$.
(A_4) $P = Q = R = 0$ has finitely many regular solutions in $\mathbf{C}_0$.
(A_5) Over a point of $\partial\mathbf{B}_0$, $P = Q = 0$ has only one solution.

Algorithm 2 `Track(`$<P,Q>$`,`$\mathbf{C}_0$`,`$\mathbf{C}_1$`,`ε`)`

Input: A system $<P,Q>$ of polynomials defining a smooth curve in an initial domain $\mathbf{C}_0 \subseteq \mathbb{R}^3$, a box $\mathbf{C}_1$ satisfying $(i),(ii)$ of Def. 1, a step-size ε.

Output: A finite sequence $\mathrm{C} = (\mathbf{C}_i)_{i=1}^m$ that is an ε-approximation of $\mathcal{C}^k$ (see Def. 1) and the looping status of $\mathcal{C}^k$.

2.2. Certified numerical path-tracking

The certified numerical path-tracker proceeds iteratively from an initial box $\mathbf{C}_1$ satisfying $(i),(ii)$ of Def. 1. Be given a sequence of boxes $(\mathbf{C}_i)_{i=1}^m$ satisfying $(i),(ii),(iii),(iv)$, it constructs a new box $\mathbf{C}_{m+1}$ connected to $\mathbf{C}_m$, ensuring $(i),(ii),(iii),(iv)$. It stops either when $\mathbf{C}_{m+1}$ contains the intersection of $\partial\mathbf{C}_0$ and $\mathcal{C}^k$ or when $m \geq 2$ and $\mathcal{C}^k \cap (\mathbf{C}_{m+1} \cup \mathbf{C}_1)$ is diffeomorphic to $[0,1]$. When the latter stopping criterion is met, $\mathcal{C}^k$ is diffeomorphic to a circle and is enclosed in the union of the boxes. When the former stopping criterion is met, one has to track $\mathcal{C}^k$ from $\mathbf{C}_1$ in the opposite way until it crosses $\partial\mathbf{C}_0$, and re-order the boxes.

Among existing approaches, [KX94, MGGJ13] describe such a numerical path tracker, [MGGJ13] uses a parametrized Krawczyk operator and paralellotopes to enclose more efficiently $\mathcal{C}$.

2.3. Connected components of $\mathcal{C}$ and initial boxes

Proposition 4 Assuming (A_1), the connected components of $\mathcal{C}$ are smooth one dimensional manifolds with or without boundary. Moreover, assuming $(A_2),(A_3)$, for any connected component $\mathcal{C}^k$ of $\mathcal{C}$, at least one of the following statements holds:

(a) $\mathcal{C}^k$ has exactly two intersections with $\partial\mathbf{C}_0$,
(b) $\mathcal{C}^k$ has at least two x-critical points.

Proof. The first part of the proposition is straightforward. One dimensional manifolds are diffeomorphic either to $]0,1[$, or to $]0,1]$, or to $[0,1]$, or to a circle. Let $\mathcal{C}^k$ be a connected component of $\mathcal{C}$. From assumption (A_2), it is compact, hence it is diffeomorphic either to $[0,1]$, or to a circle. Suppose first $\mathcal{C}^k$ has an intersection with $\partial\mathbf{C}_0$. From (A_3), this intersection is a point, hence $\mathcal{C}^k$ is diffeomorphic to $[0,1]$ and has a second intersection with $\partial\mathbf{C}_0$. Suppose now that $\mathcal{C}^k$ does not intersect $\partial\mathbf{C}_0$. Hence it is diffeomorphic to a circle and, since it is compact, it has minimum and maximum x-coordinates. Assertion (b) follows. □

As a direct consequence of Prop. 4, the following corollary gives a constructive characterization of a set containing at least one point on each connected component of $\mathcal{C}$.

Corollary 5 Consider the following systems of equations:

(S_1) $P(l(\mathbf{x}_0), y, z) = Q(l(\mathbf{x}_0), y, z) = 0$,
(S_2) $P(u(\mathbf{x}_0), y, z) = Q(u(\mathbf{x}_0), y, z) = 0$,
(S_3) $P(x, l(\mathbf{y}_0), z) = Q(x, l(\mathbf{y}_0), z) = 0$,
(S_4) $P(x, u(\mathbf{y}_0), z) = Q(x, u(\mathbf{y}_0), z) = 0$,
(S_5) $P(x, y, z) = Q(x, y, z) = R(x, y, z) = 0$.

Assuming $(A_1),(A_2),(A_3),(A_4)$, the set of solutions of $S_1 \vee$

Algorithm 3 Enclose $\mathcal{C}$

Input: A box $\mathbf{B}_0$ in $\mathbb{R}^2$, two polynomials P and Q in $\mathbb{Q}[x,y,z]$, a step-size ε.
Output: Boxes $(\mathbf{C}_i)_{1\leq i\leq m}$ enclosing $\mathcal{C}$ and the sequence $(s_k)_{1\leq k\leq n}$ of first and last indexes corresponding to connected components of $\mathcal{C}$

1: $\mathrm{C}^b \leftarrow$ `IsolateSols(`$S_1\vee\ldots\vee S_4$`,`$\partial(\mathbf{B}_0\times\mathbb{R})$`)`
2: $\mathrm{C}^x \leftarrow$ `IsolateSols(`S_5`,`$\mathbf{B}_0\times\mathbb{R}$`)`
3: $\mathcal{L} = \mathrm{C}^b \cup \mathrm{C}^x$
4: $n=0$, $b_1=1$.
5: **while** $\mathcal{L}\neq\emptyset$ **do**
6: $\quad n=n+1$
7: $\quad \mathbf{C}\leftarrow pop(\mathcal{L})$
8: $\quad (\mathbf{C}_j)_{b_n\leq j\leq e_n} \leftarrow$ `Track(`P,Q`,`$\mathbf{B}_0\times\mathbb{R}$`,`$\mathbf{C}$`,`$\varepsilon$`)`
9: $\quad$ **for** j s.t. $b_n\leq j\leq e_n]$ **do**
10: $\quad\quad$ **for** $\mathbf{C}'\in\mathcal{L}$ s.t. $\mathbf{C}'\cap\mathbf{C}_j\neq\emptyset$ **do**
11: $\quad\quad\quad$ Contract $\mathbf{C}'$ until $\mathbf{C}'\cap\mathbf{C}_j=\emptyset$ or $\mathbf{C}'\subset\mathbf{C}_j$
12: $\quad\quad\quad$ **if** $\mathbf{C}'\subset\mathbf{C}_j$ **then** Remove $\mathbf{C}'$ from $\mathcal{L}$
13: $\quad$ **if** $(\mathbf{C}_j)_{b_n\leq j\leq e_n}$ encloses a loop **then**
14: $\quad\quad s_n=(b_n,b_n)$
15: $\quad$ **else**
16: $\quad\quad s_n=(b_n,e_n)$
17: $\quad b_{n+1}=e_n+1$
18: $m=e_n$
19: **return** $(\mathbf{C}_i)_{1\leq i\leq m}$ and $(s_k)_{1\leq k\leq n}$.

$\ldots\vee S_5$ is a finite set of points, containing at least one point on each connected component of $\mathcal{C}$.

As a consequence, it is possible to construct a set containing at least one point on each connected component of $\mathcal{C}$ by isolating solutions of $S_1\vee\ldots\vee S_4$ in the domain $\partial\mathbf{C}_0$, and S_5 in the domain $\mathbf{C}_0$.

2.4. One dimensional solver

The way we process to enclose $\mathcal{C}$ in a sequence $(\mathbf{C}_i)_{1\leq i\leq m}$ of boxes of $\mathbb{R}^3$ is detailed in Algo. 3. Let $\mathcal{C}^1,\ldots,\mathcal{C}^n$ be the connected components of $\mathcal{C}$. Each $\mathcal{C}^k$ will be enclosed in a sub-sequence $(\mathbf{C}_i)_{b_k\leq i\leq e_k}$, with $1\leq b_k<e_k\leq m$.

In steps 1, 2 and 3 of Algo. 3, a set $\mathcal{L}$ containing at least one point on each connected component of $\mathcal{C}$ is computed with the procedure `IsolateSols(,)`. It is assumed in Algo. 3 that boxes obtained in this step satisfies properties $(i),(ii)$ of Def. 1. Otherwise they are contracted with the appropriate contractor K_{S_i} until this is the case. Then connected components are tracked from points of $\mathcal{L}$ using the procedure `Track(.,.,.,.)`.

To avoid redondancy in $(\mathbf{C}_i)_{1\leq i\leq m}$, that is several sub-sequences enclosing the same connected component, all boxes in $\mathcal{L}$ containing a point of $\mathcal{C}^k$ have to be removed from $\mathcal{L}$ when a connected component $\mathcal{C}^k$ is enclosed in $(\mathbf{C}_i)_{b_k\leq i\leq e_k}$. It is achieved in steps 9 to 12. Notice first that if a point p of $\mathcal{C}$ is isolated in $\mathbf{C}'$ and if there exists $b_k\leq j\leq e_k$ s.t. $\mathbf{C}'\subset\mathbf{C}_j$, then $p\in\mathcal{C}^k$ (otherwise p belongs to an other connected component $\mathcal{C}^l$ intersecting $\mathbf{C}_j$, and $\mathcal{C}\cap\mathbf{C}_j$ is not diffeomorphic to an open segment as required by Def. 1). Now, if there exist $b_k\leq j\leq e_k$ and $\mathbf{C}'\in\mathcal{L}$ s.t. $\mathbf{C}'\cap\mathbf{C}_j\neq 0$, since $\mathbf{C}$ has been obtained with the procedure `IsolateSols(.,.)`, it can be contracted with the appropriate operator until $\mathbf{C}'\cap\mathbf{C}_j=\emptyset$ or $\mathbf{C}'\subset\mathbf{C}_j$, which allows to decide if p belongs to $\mathcal{C}^k\cap\mathbf{C}_j$ (we assume here that p does not belong to $\partial\mathbf{C}_j$, otherwise $\mathcal{C}^k$ has to be re-enclosed with a smaller step-size).

Finally, the pairs (b_k,e_k) of indices of the first and last box enclosing the connected components $\mathcal{C}^k$ are stored in a list of pairs $(s_k)_{k=1}^n$. If $\mathcal{C}^k$ is diffeomorphic to a loop, then we store the pair (b_k,b_k).

By construction, the sequence $(\mathbf{C}_i)_{b_k\leq i\leq e_k}$ is an ε-approximation of $\mathcal{C}^k$ that doesn't overlap any other ε-approximation of a component $\mathcal{C}^j$ for $j\neq k$.

3. Isolating singularities of $\mathcal{B}$

When $\mathcal{C}$ is defined by two analytic maps P,Q, [IMP15a] describes, under genericity conditions on P,Q, the type of singularities arising in the projection $\mathcal{B}$: they are only nodes (two branches of $\mathcal{C}$ induce a self intersection in $\mathcal{B}$), or cusps ($\mathcal{C}$ has a vertical tangent). [IMP15a] also introduces a system called *ball system* which solutions are singularities of $\mathcal{B}$, and shows that the ball system admits only regular solutions if and only if singularities of $\mathcal{B}$ are either nodes or ordinary cusps.

We first restate the assumptions and the main results of [IMP15a] in the more restrictive case where P,Q are polynomials. Then we show how an enclosure of $\mathcal{C}$ and $\mathcal{B}$ can help to restrict the domain where the ball system is solved while ensuring that all points of Σ are obtained. Then we present Algo. 4 that decides, for a given solution of the ball system, if the corresponding singularity is an ordinary cusp or a node when $\mathcal{B}$ is an apparent contour.

3.1. Assumptions

Consider the following assumptions:

(A_6) For any (α,β) in $\mathbf{B}_0$, the system $P(\alpha,\beta,z)=Q(\alpha,\beta,z)=0$ has at most 2 real roots counted with multiplicities.
(A_7) There is finitely many points (α,β) in $\mathbf{B}_0$ such that $P(\alpha,\beta,z)=Q(\alpha,\beta,z)=0$ has 2 real roots counted with multiplicities.
(A_8) $\pi_{(x,y)}$ restricted to the curve $\mathcal{C}$ is a proper map, that is the inverse image of a compact is compact.
(A_9) The singularities of the curve $\mathcal{B}$ in $\mathbf{B}_0$ are either nodes or ordinary cusps.

Assumption (A_2) given in Sec. 2.1 is a consequence of (A_8). Notice that Thom Transversality Theorem implies that $(A_1),(A_6),\ldots,(A_9)$ hold for generic polynomial maps P,Q defining $\mathcal{C}$ (see [Dem00, Th. 3.9.7 and §4.7]).

3.2. Ball system

Following a geometric modelisation, [IMP15a] defines a 4 dimensional system which solutions maps to the singularities of $\mathcal{B}$. In this modelisation, two solutions (x,y,z_1) and (x,y,z_2) of $P=Q=0$ (or $P=P_z=0$) are mapped to the point (x,y,c,r_2) with $c=(z_1+z_2)/2$ and $r_2=(z_1-z_2)^2$.

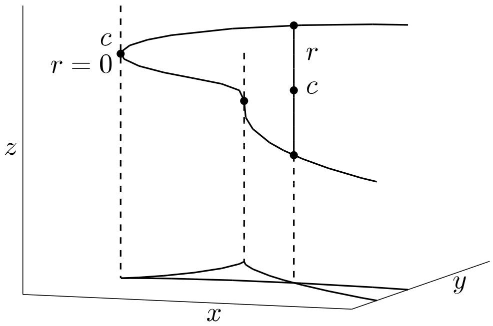

Figure 3: *Singularities of the apparent contour of the torus. For nodes and cusps singularities, their antecedents as well as corresponding centers and radius are represented.*

Fig. 3 illustrates this mapping for singularities of the apparent contour of a torus.

We recall the main results of [IMP15a] that were stated for analytic maps P,Q.

Lemma 6 ([IMP15a]) Let P,Q be two polynomials in $\mathbb{Q}[x,y,z]$, and $\mathcal{S}$ be the set of solutions of the so-called *ball system*:

$$\left\{\begin{array}{r} \frac{1}{2}(P(x,y,c+\sqrt{r_2})+P(x,y,c-\sqrt{r_2}))=0 \\ \frac{1}{2\sqrt{r_2}}(P(x,y,c+\sqrt{r_2})-P(x,y,c-\sqrt{r_2}))=0 \\ \frac{1}{2}(Q(x,y,c+\sqrt{r_2})+Q(x,y,c-\sqrt{r_2}))=0 \\ \frac{1}{2\sqrt{r_2}}(Q(x,y,c+\sqrt{r_2})-Q(x,y,c-\sqrt{r_2}))=0 \end{array}\right. \tag{1}$$

in $\mathbf{B}_0 \times \mathbb{R} \times \mathbb{R}^+$. Then $\pi'_{(x,y)}(\mathcal{S}) = \Sigma$, where $\pi'_{(x,y)}$ is the projection from $\mathbb{R}^4$ to the (x,y)-plane.

Lemma 7 ([IMP15a]) Under the Assumptions $(A_1),(A_6)-(A_8)$, all the solutions of the ball system in $\mathbf{B}_0 \times \mathbb{R} \times \mathbb{R}^+$ are regular if and only if (A_9) is satisfied.

3.3. Solving domain

In [IMP15a], the ball system is solved within the box $\mathbf{B}_0 \times \mathbb{R} \times \mathbb{R}^+$ thanks to a subdivision solver to isolate all the singularities of $\mathcal{B}$ in $\mathbf{B}_0$.

Here, in a first step presented below, we enclose $\mathcal{C}$ in a sequence of small boxes with a certified path tracking algorithm. The output of the latter algorithm is an ε-approximation of $\mathcal{C}$. Given a singular point σ of $\mathcal{B}$, there exists $1 \le i \le m$ such that $\sigma \in \mathbf{B}_i$, where $\mathbf{B}_i = \pi_{(x,y)}(\mathbf{C}_i)$. Hence it is possible to isolate all singularities by solving the ball system within $\mathbf{B}_i \times \mathbb{R} \times \mathbb{R}^+$, for $1 \le i \le m$.

In addition, the enclosure $(\mathbf{C}_i)_{1\le i\le m}$ of $\mathcal{C}$ allows us to bound the solving domain in c and r_2 components.

Proposition 8 Suppose that for $1 \le i \le m$, $\mathbf{C}_i = (\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i)$ (hence $\mathbf{B}_i = (\mathbf{x}_i, \mathbf{y}_i)$), note $\mathbf{B}_{ij} = (\mathbf{x}_{ij}, \mathbf{y}_{ij}) = \mathbf{B}_i \cap \mathbf{B}_j$ for $1 \le i < j \le m$, and consider the sets:

- $\mathbf{D}_i = (\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i, [0, (\frac{w(\mathbf{z}_i)}{2})^2])$,
- $\mathbf{D}_{ij} = (\mathbf{x}_{ij}, \mathbf{y}_{ij}, \frac{(\mathbf{z}_i+\mathbf{z}_j)}{2}, (\frac{(\mathbf{z}_i-\mathbf{z}_j)}{2})^2)$

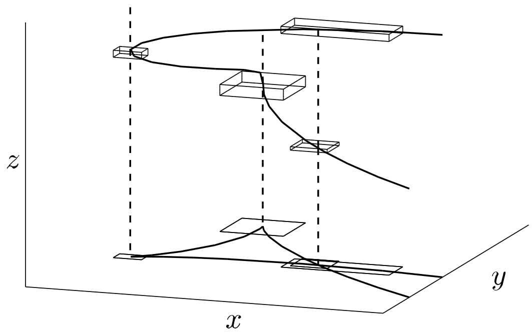

Figure 4: *Some boxes and their projections containing singularities of $\mathcal{B}$. Cusps singularities are in boxes $\mathbf{B}_i$, nodes in boxes $\mathbf{B}_i \cap \mathbf{B}_j$.*

Then all solutions of the ball system lie in $(\bigcup_{1\le i\le m} \mathbf{D}_i) \cup (\bigcup_{1\le i<j\le m} \mathbf{D}_{ij})$.

Proof. Let $p = (x_p, y_p, c_p, r_p) \in \mathbf{B}_0 \times \mathbb{R} \times \mathbb{R}^+$ be a solution of the ball system and $\sigma = \pi'_{(x,y)}(p)$ the corresponding singularity in Σ. From Assumption (A_9), σ is either an ordinary cusp or a node.

Suppose first it is an ordinary cusp. Then $r_p = 0$, and σ is the projection of a single point $p' = (x_p, y_p, c_p)$ of $\mathcal{C}$. Hence there exists $1 \le i \le m$ such that $p' \in \mathbf{C}_i$. As a consequence we have $c_p \in \mathbf{z}_i$ and $p \in \mathbf{D}_i$. See Fig. 4.

Suppose now σ is a node. Then $r_p > 0$, and σ is the projection of two points $p_- = (x_p, y_p, c_p - \sqrt{r_p})$ and $p_+ = (x_p, y_p, c_p + \sqrt{r_p})$ of $\mathcal{C}$. Hence there exist $1 \le i \le m$ and $1 \le j \le m$ such that $p_- \in \mathbf{C}_i$ and $p_+ \in \mathbf{C}_j$. If $i = j$, we have $c_p \in \mathbf{z}_i$ and $r_p \in [0, (\frac{w(\mathbf{z}_i)}{2})^2]$, and finally $p \in \mathbf{D}_i$. If $i \neq j$ (this case is illustrated in Fig. 4), c_p lies in $\frac{\mathbf{z}_i+\mathbf{z}_j}{2}$ that is the center of the two intervals $\mathbf{z}_i$ and $\mathbf{z}_j$, and r_p lies in $(\frac{\mathbf{z}_i-\mathbf{z}_j}{2})^2$, that is the square or the corresponding ray. $\square$

3.4. Singularites of an apparent contour

When $\mathcal{B}$ is defined as the apparent contour of a smooth algebraic surface, its singularities can be ordinary cusps or nodes, and it is necessary to distinguish these two types of singularities.

Let $\mathbf{D} = (\mathbf{x}, \mathbf{y}, \mathbf{c}, \mathbf{r})$ be a box isolating a solution $p = (x,y,c,r) \in \mathbf{D}$ of the ball system. If $0 \notin \mathbf{r}$ then $r \neq 0$ and $\pi'_{(x,y)}(p)$ is a node. Otherwise, $\pi'_{(x,y)}(p)$ can be either a cusp or a node.

Recall now that $\sigma \in \Sigma$ is an ordinary cusp of $\mathcal{B}$ only if it is the projection of a point of $\mathcal{C}_{P\cap P_z}$ that has a vertical tangent. Formally speaking, if $\sigma = (\alpha, \beta)$ is a cusp, there exists a unique $\gamma \in \mathbb{R}$ such that $\sigma = \pi_{(x,y)}(\alpha, \beta, \gamma)$ and $P(\alpha,\beta,\gamma) = P_z(\alpha,\beta,\gamma) = P_{zz}(\alpha,\beta,\gamma) = 0$. Hence, making the assumption that the system $P = P_z = P_{zz} = 0$ has only regular solutions and noting $K_{(P,P_z,P_{zz})}$ the Krawczik operator for the latter system, there exists a box $\mathbf{C}_\sigma$ containing (α,β,γ) such that $K_{(P,P_z,P_{zz})}(\mathbf{C}_\sigma) \subset int(\mathbf{C}_\sigma)$.

Algorithm 4 Singularity types for an apparent contour

Input: P in $\mathbb{Q}[x,y,z]$, a box $\mathbf{D} = (\mathbf{x},\mathbf{y},\mathbf{c},\mathbf{r})$ s.t $K_b(\mathbf{D}) \subset int(\mathbf{D})$.
Output: The type of the singularity contained in $\pi'_{(x,y)}(\mathbf{D}) = (\mathbf{x},\mathbf{y})$.
1: **while** $0 \in \mathbf{r}$ **do**
2: **if** $K_{(P,P_z,P_{zz})}((\mathbf{x},\mathbf{y},\mathbf{c})) \subset int((\mathbf{x},\mathbf{y},\mathbf{c}))$ **then return** cusp
3: $\mathbf{D} = K_b(\mathbf{D})$
4: **return** node

As a consequence, there exists a box $\mathbf{D}_* = (\mathbf{x}_*,\mathbf{y}_*,\mathbf{c}_*,\mathbf{r}_*) \subset \mathbf{D}$, containing p such that either $0 \notin \mathbf{r}_*$ and $\pi'_{(x,y)}(p)$ is a node, or $K_{(P,P_z,P_{zz})}((\mathbf{x}_*,\mathbf{y}_*,\mathbf{c}_*)) \subset int((\mathbf{x}_*,\mathbf{y}_*,\mathbf{c}_*))$ and $\pi'_{(x,y)}(p)$ is an ordinary cusp.

Algo. 4 formalizes this decision procedure for a box $\mathbf{D}$ containing a unique solution of the ball system. It terminates since we have $K_b(\mathbf{D}) \subset int(\mathbf{D})$ only if the sequence $\mathbf{D}_1 = \mathbf{D}$, $\mathbf{D}_i = K_b(\mathbf{D}_{i-1})$ converges quadratically, where K_b denotes the Krawczik operator for the ball system.

4. Global topology of $\mathcal{B}$ as a non-embedded graph

This section proposes an encoding of the topology of the projected curve $\mathcal{B}$ as a graph $G_\mathcal{B}$. We consider on a graph the natural topology, as defined in [GT87, §1.3.2, p. 18], where each vertex is represented by a distinct point and each edge by a distinct arc homeomorphic to the closed interval $[0,1]$. Our aim is thus to define $G_\mathcal{B}$ such that it is homeomorphic to $\mathcal{B}$. In addition, we also want to identify each cusp as a vertex of the graph, which is done by adding labels.

The graph $G_\mathcal{B}$ associated to a projection or an apparent contour $\mathcal{B}$ is constructed as follows. First, the curve $\mathcal{C}$ is enclosed in a sequence of boxes $(\mathbf{C}_i)_{1\le i\le m}$ with Algo. 3, and a graph $G_\mathcal{C}$ encoding the connections of boxes enclosing $\mathcal{C}$ is constructed. Using the properties of the output of Algo. 3, the graph $G_\mathcal{C}$ is thus homeomorphic to the curve $\mathcal{C}$. Second, boxes or pairs of boxes in $(\mathbf{C}_i)_{1\le i\le m}$ are identified as defining cusps and nodes in the projection to $\mathcal{B}$. Finally, the graph $G_\mathcal{B}$ is constructed from $G_\mathcal{C}$ by merging vertices for nodes and adding labels for cusps.

4.1. Computing connected components of $\mathcal{C}$

In a first step, the curve $\mathcal{C}$ is enclosed in a sequence of boxes thanks to Algo. 3. The output of the latter is the sequence of boxes $(\mathbf{C}_i)_{1\le i\le m}$ and for each connected component $\mathcal{C}^k$ of $\mathcal{C}$, the bounding indices (b_k,e_k) of boxes of $(\mathbf{C}_i)_{1\le i\le m}$ enclosing $\mathcal{C}^k$. Recall that for a connected component $\mathcal{C}^k$ that is diffeomorphic to a circle, $e_k = b_k$, and the last box enclosing $\mathcal{C}^k$ is given by the index $b_{k+1}-1$, with b_{n+1} defined by $b_{n+1} = m$.

The graph $G_\mathcal{C} = (V_\mathcal{C},E_\mathcal{C})$ together with the function $e_\mathcal{C} : E_\mathcal{C} \to V_\mathcal{C} \times V_\mathcal{C}$ encoding the connection of boxes of $(\mathbf{C}_i)_{1\le i\le m}$ is constructed as follows. $V_\mathcal{C}$ contains a vertex v_i for each box $\mathbf{C}_i$. For each connected component $\mathcal{C}^k$, each consecutive pair of boxes define an edge of $E_\mathcal{C}$. More precisely, for a component $\mathcal{C}^k$ that is diffeomorphic to a segment, for each i such that $b_k \le i < e_k$ an edge which image by the map $e_\mathcal{C}$ is (v_i,v_{i+1}) is defined. For a component $\mathcal{C}^k$ that is diffeomorphic to a circle, for each i such that $b_k \le i < b_{k+1}-1$ an edge which image by the map $e_\mathcal{C}$ is (v_i,v_{i+1}) is defined, in addition, the edge which image is $(v_{b_k},v_{b_{k+1}-1})$ is also defined. According to the properties of the output of Algo. 3 (see Sec. 2), the graph $G_\mathcal{C}$ is thus homeomorphic to the curve $\mathcal{C}$. Note that $G_\mathcal{C}$ does not need to be labeled, since each point of $\mathcal{C}$ is regular.

4.2. Computing the graph $G_\mathcal{B}$

Let $1 \le i < j \le m$, and boxes $\mathbf{D}_i$ and $\mathbf{D}_{ij}$ be defined as in Prop. 8 from the enclosure of $\mathcal{C}$ by the sequence $(\mathbf{C}_i)_{1\le i\le m}$. We define the following conditions on these 3 or 4 dimensional boxes.

- (c_1) Each solution of the ball system that maps to a cusp by projection is in a unique $\mathbf{D}_i$.
- (c_2) Each solution of the ball system that maps to a node by projection is in a unique $\mathbf{D}_{ij}$ and such a solution is not in any $\mathbf{D}_i$.
- (c_1) The ball system has no more than one solution in $\mathbf{D}_i$ and $\mathbf{D}_{ij}$.
- (c_3) Suppose the ball systems has a solution that projects to a node in $\mathbf{D}_{ij}$. Then if $l \ne i,j$, the ball systems does not admit a solution in $\mathbf{D}_{li}$, $\mathbf{D}_{il}$, $\mathbf{D}_{lj}$ or $\mathbf{D}_{jl}$.
- (c_4) Suppose the ball systems has a solution in $\mathbf{D}_{ij}$. Then $\mathbf{C}_i,\mathbf{C}_j$ are not the first or last boxes of the enclosure of some connected component diffeomorphic to a segment.

Subsec. 4.3 describes how to achieve these conditions by refining the tracking of the curve. In the following, we assume these conditions are satisfied and focus on the construction of the graph $G_\mathcal{B}$ from the graph $G_\mathcal{C}$ and this data.

Let $I^n = \{(i_1,j_1),\dots,(i_s,j_s)\}$ be the set of pairs of indexes of boxes $\mathbf{D}_{i_l,j_l}$ containing the solutions of the ball system projecting to nodes. The non-labeled graph $G_\mathcal{B} = (V_\mathcal{B},E_\mathcal{B})$ is such that $V_\mathcal{B} = V_\mathcal{C} \setminus \{v_{j_1},\dots,v_{j_s}\}$ and $E_\mathcal{B} = E_\mathcal{C}$. The map $e_\mathcal{B} : E_\mathcal{B} \to V_\mathcal{B} \times V_\mathcal{B}$ is defined for an edge e as follows: if there exists $(i_k,j_k) \in I^n$ such that $e_\mathcal{C}(e) = (v_{j_k},v_l)$ then $e_\mathcal{B}(e) = (v_{i_k},v_l)$, else $e_\mathcal{B}(e) = e_\mathcal{C}(e)$.

In addition, we define labels on the vertices as a function $\gamma_{\text{type}} : V_\mathcal{B} \to L$ with $L = \{\text{regular},\text{cusp},\text{node}\}$. For a box $\mathbf{D}_i$ containing a solution of the ball system projecting to a cusp, $\gamma_{type}(v_i) = \text{cusp}$; for any $(i_k,j_k) \in I^n$, $\gamma_{type}(v_{i_k}) = \text{node}$; for all other vertices v, $\gamma_{type}(v) = \text{regular}$.

Proposition 9 The graph labeled graph $G_\mathcal{B} = (V_\mathcal{B},E_\mathcal{B},L)$ with applications $e_\mathcal{B}$ and γ_{type} is homeomorphic to the curve $\mathcal{B}$ in the sense of Def. 2.

Proof From the previous section, the graph $G_\mathcal{C}$ is homeomorphic to the curve $\mathcal{C}$. The changes of topology between the curve $\mathcal{C}$ and its projection $\mathcal{B}$ only occur at nodes. At a node, two points of $\mathcal{C}$ project to a single point of $\mathcal{B}$ which is then connected to 4 branches of curve. Using the graphs, this change of topology is performed by merging two vertices of $G_\mathcal{C}$ in a single vertex of $G_\mathcal{B}$. Condition (c_2) implies that all nodes are uniquely identified as a pair of vertices of

$G_{\mathcal{C}}$. Condition (c_3) implies that any vertex of $G_{\mathcal{C}}$ appears in at most one such pair, and condition (c_4) ensures that such a vertex is not a vertex of degree 1. Thus each merging involves of a pair of vertices of degree 2 creating a vertex of degree 4 in $G_{\mathcal{B}}$ as expected. Finally, it is clear that the labels satisfy the condition of Def. 2. □

4.3. Checking conditions (c_i) of Sec. 4.2

The aim of this section is to prove the following proposition.

Proposition 10 Under assumption (A_1) to (A_9) there exists a step-size ε such that the enclosure obtained as the output of Algo. 3 satisfies the conditions $(c_1), \dots, (c_4)$ of Sec. 4.2.

We state the following Lemma in order to prove Prop. 10.

Lemma 11 Let p_1 and p_2 be distinct points on $\mathcal{C}$. Then there exists a step-size ε such that the enclosure $(\mathbf{C}_i)_{1\leq i\leq m}$ obtained as the output of Algo. 3 satisfies that a box containing p_1 does not contain p_2.

Proof of Lem. 11. The distance between two points in a box of width w is at most $\sqrt{3}w$, thus it is enough to require $\varepsilon < d/\sqrt{3}$, where d is the distance between p_1 and p_2. □

Proof of Prop. 10. The proof is based on a recursive algorithm that has as input an enclosure of $\mathcal{C}$ by the boxes $\mathbf{C}_i$ output by Algo. 3. It defines a sequence of checks on the boxes $\mathbf{C}_i$, $\mathbf{D}_i$ or $\mathbf{D}_{ij}$ to be performed such that if any of these checks fails then the current step-size is divided by 2, Algo. 3 is performed with this new step-size and the sequence of checks is performed on this new enclosure.

We first check that each box $\mathbf{D}_i$ or $\mathbf{D}_{ij}$ has at most one solution of the ball system. This can be checked using the procedure `IsolateSols(.,.)`. Algo. 4 is then able to determine the type node or cusp of the associate projection of the solution. We discard the boxes $\mathbf{D}_{ij}$ containing a cusp since this solution is also reported in the boxes $\mathbf{D}_i$ or $\mathbf{D}_j$. We also check that a box $\mathbf{D}_i$ does not contain a node, this will eventually be possible since the two points of the curve $\mathcal{C}$ projecting to the node can be separated in different boxes $\mathbf{C}_j$.

Still some solution can be reported several times in different boxes. For cusps, we check that for two boxes $\mathbf{D}_i$ and $\mathbf{D}_j$ each containing a cusp, their intersection does not contain a solution of the ball system (using the procedure `IsolateSols(.,.)`). Similarly, for nodes, we check that for two boxes $\mathbf{D}_{ij}$ and $\mathbf{D}_{kl}$ (i,j,k,l distinct) each containing a node, their intersection does not contain a solution of the ball system. A this step, we then have the conditions (c_1) and (c_2) satisfied. We then check (c_3), it will eventually be satisfied since the nodes reported by $\mathbf{D}_{ij}$ and $\mathbf{D}_{ik}$ being distinct, they are correspond to 4 distinct points on the curve $\mathcal{C}$ that will be separated for a small enough step-size. We finally check (c_4) which will eventually be satisfied since by assumption (A_5) a node cannot be on the boundary. □

As suggested by Prop. 10, conditions $(c_1), \dots, (c_4)$ can be checked on an enclosure $(\mathbf{C}_i)_{1\leq i\leq m}$ obtained with a step-size ε once the ball system has been solved in $(\bigcup_{1\leq i\leq m}\mathbf{D}_i)\cup(\bigcup_{1\leq i<j\leq m}\mathbf{D}_{ij})$. While $(c_1), \dots, (c_4)$ are not satisfied, one has to take, say, $\varepsilon' = \frac{\varepsilon}{2}$, and to compute a new enclosure with step-size ε', to solve once again the ball system.

In a more practical approach conditions $(c_1), \dots, (c_4)$ can be checked during the computation of an enclosure. Each time a box $\mathbf{C}_i$ is obtained, $\mathbf{D}_i$ and $\mathbf{D}_{hi}$ for $1 \leq h < i$ are computed to solve the ball system in $\mathbf{D}_i \bigcup_{h=1}^{i-1}\mathbf{D}_{hi}$. While $(c_1), \dots, (c_4)$ are not satisfied, either $\mathbf{C}_i$ can be refined by enclosing $\mathcal{C}$ in the box $\mathbf{C}_i$ with a step-size $\frac{\varepsilon}{2}$ (with procedure `Track(.,.,.,.)`) or the appropriated box $\mathbf{C}_h$ with $h < i$ can be refined.

5. Implementation and results

We did implement the major part of the approach presented here, more precisely the Algo. 3, consisting in enclosing a curve $\mathcal{C}$, and the isolation of the singularities of its projection $\mathcal{B}$ with the ball system, as described in Prop 8.

Subsection 5.1 presents implementation of important procedures that are `IsolateSols(.,.)`, and `Track(.,.,.,.)`. Subsection 5.2 presents results obtained with the approach proposed here about isolating the singularities of an apparent contour, and compare it with other approaches.

5.1. Implementation

The procedure `IsolateSols(.,.)` is implemented by a subdivision solver, written in `c++`, handling arbitrary arithmetic precision. It uses the `boost` interval library when working in double precision, and the `mpfi` interval library if arbitrary precision in needed. Polynomials are evaluated at order 2 with Horner scheme to make the interval evaluations quick and sharp. The Krawczyk operator at order 2 is used to certify existence and uniqueness or absence of a solution in a box. We did interface the subdivision solver with the mathematical software `sage` via `cython`. The implementation is described with more details in [IMP15b].

The procedure `Track(.,.,.,.)` is a prototype implementation in `python` of the approach described in [MGGJ13]. It is doomed to be translated in `c++`.

5.2. Results

We present here results about isolating singularities of an apparent contour or a surface P with the approach proposed in this paper, and we compare it with two other approaches.

Experimental data are random dense polynomial P generated with degree d and integer coefficients chosen uniformly in $[\![-2^8, 2^8]\!]$. Singularities of apparent contours are isolated in $\mathbf{B}_0 = [-1,1]\times[-1,1]$. The given running times are averages over five instances for a given degree d and have to be understood as sequential times in seconds.

Testing environment is an Intel(R) Xeon(R) CPU L5640 @ 2.27GHz machine with Linux.

Table 1: *Running times of four approaches to isolate singularities of apparent contour. Input polynomials have degree d and $\mathbf{B}_0 = [-1,1] \times [-1,1]$. The running times are in seconds. (1):* `RSCube` *and system $R = R_x = R_y = 0$ where R is a resultant; (2):* `IsolateSols(.,.)` *and sub-resultant system; (3):* `IsolateSols(.,.)` *and ball system; (4): method presented here.*

domain d	(1) $\mathbb{R}^2$	(2) $\mathbf{B_0}$	(3) $\mathbf{B_0}$	(4) $\mathbf{B_0}$
5	3.1	0.05	24.8	1.25
6	32	0.50	8.40	2.36
7	254	4.44	43.8	4.13
8	1898	37.9	70.2	5.91
9	9346	23.1	45.6	5.30

Considered approaches. Singularities of a plane curve $\mathcal{B}$ defined by a polynomial $R \in \mathbb{Q}[x,y]$ are classically characterized as the solutions of the system of equations $R = R_x = R_y = 0$. This system is not squared and does not suit to numerical solving. Efficient bivariate solver such as `RSCube` (a `Maple` package that uses triangular decompositions and Rational Univariate Representations, see [BLPR11, Bou14]) are able to handle such systems. Here $\mathcal{B}$ is defined as a projection, and the polynomial R can be defined as the resultant of P, P_z. However the resultant is a large polynomial in terms of degree, bit-size, and number of coefficients. Here singularities of $\mathcal{B}$ are isolated in $\mathbb{R}^2$. Column (1) of table 1 reports results obtained with this approach.

Thanks to coefficients of the sub-resultant chain, [IMP15b] characterizes singularities of a projection or an apparent contour $\mathcal{B}$ as the regular solutions of a system in two equations involving two unknowns. This system can be solved with the procedure `IsolateSols(.,.)`, and column (2) of table 1 gives times needed to solve this system.

The ball system, introduced in [IMP15a], can be solved with the procedure `IsolateSols(.,.)` within $\mathbf{B}_0 \times \mathbb{R} \times \mathbb{R}^+$. Column (3) of table 1 reports time of isolation.

Finally, the column (4) shows times of the whole process (enclosing $\mathcal{C}$ and isolating the singularities) presented in this paper to isolate singularities of an apparent contour.

Table 1 shows results obtained when using each method described above to isolate singularities of the apparent contour.

- The first approach suffer from the size of resultant polynomials. For instance, when p is a dense polynomial of degree 6, the resultant of P, P_z has degree 30 and 495 monomials which coefficients have a bit-size 111.
- When using a subdivision solver such as the one implemented by `IsolateSols(.,.)`, running times are subject to an important variance, this explains that several results may appear surprising (see columns (3), $d = 5, 6$).
- Even if the polynomials involved in the sub-resultant system are large, the resolution of the latter system is more efficient than the resolution of the ball system within $\mathbf{B}_0 \times \mathbb{R} \times \mathbb{R}^+$.
- Enclosing $\mathcal{C}$ with a certified numerical path tracker restricts the domain where the ball system is solved and thus speeds up the isolation time.

6. Conclusion

This paper presents a method to compute both geometrical and topological information about a plane curve $\mathcal{B}$ defined as the projection of a space curve $\mathcal{C}$ or as the apparent contour of a surface.

The geometry of the plane curve in a bounded domain of the plane is computed thanks to a certified numerical path tracker applied to the space curve. Singularities induced by the projection are characterized as nodes or ordinary cusps, and are isolated with a certified numerical solving method. Then the topology of the plane curve is represented by a non-embedded graph. Combinatorial information carried by such graphs are sufficient to decide if two plane curves are homeomorphic by comparing their associated graphs.

We experimented our approach and got promising results: local topological informations such as singularities are isolated more efficiently than with recent methods.

We will now focus on employing tools presented here to compute a stronger representation, with an embedded graph, of a projected curve and/or an apparent contour. Such embedded graphs describe the ambient homeomorphism class of a curve: with simple words, they carry informations about the faces of the subdivision of the considered domain induced by the plane curve.

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