Numeric and Certified Isolation of the Singularities of the Projection of a Smooth Space Curve

Rémi Imbach, Guillaume Moroz and Marc Pouget

Vegas
Visibility Surfaces

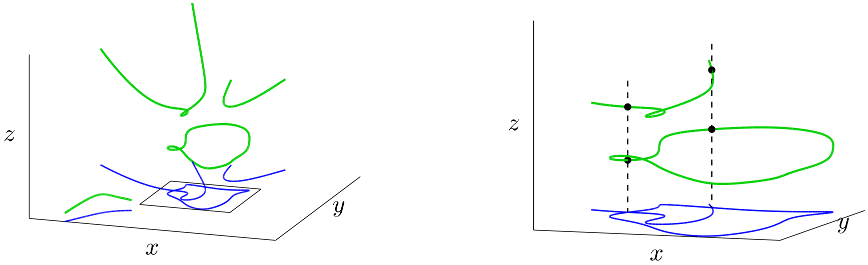
Projection and Apparent Contour

P, Q two analytic maps $\mathbb{R}^3 \to \mathbb{R}$

Curve defined as the intersection of two surfaces:

$$\mathcal{C} : \begin{cases} P(x,y,z) &= 0 \\ Q(x,y,z) &= 0 \end{cases}, (x,y,z) \in \mathbb{R}^3$$

Projection in the plane: $\pi_{(x,y)}(\mathcal{C})$



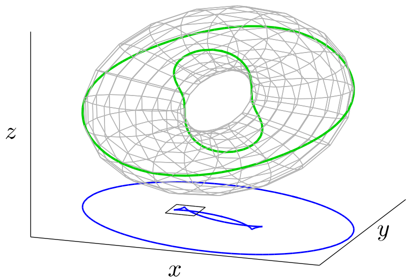
Projection and Apparent Contour

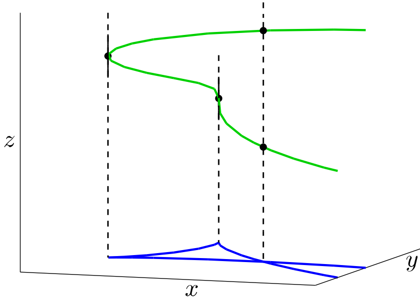
P, Q two analytic maps $\mathbb{R}^3 \rightarrow \mathbb{R}$

Curve defined as the intersection of two surfaces:

$$\mathcal{C} : \begin{cases} P(x,y,z) &= 0 \\ P_z(x,y,z) &= 0 \end{cases}, (x,y,z) \in \mathbb{R}^3, \qquad P_z = \frac{\partial P}{\partial z}$$

Apparent contour: $\pi_{(x,y)}(\mathcal{C})$





Problematic

P, Q two analytic maps $\mathbb{R}^3 \to \mathbb{R}$, possibly $Q = P_z = \frac{\partial P}{\partial z}$

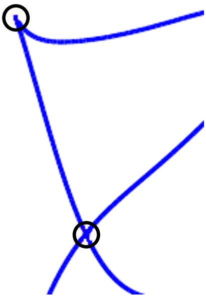
Curve defined as the intersection of two surfaces:

$$\mathcal{C} : \left\{ \begin{array}{ll} P(x,y,z) & = 0 \\ Q(x,y,z) & = 0 \end{array} \right. , (x,y,z) \in \mathbb{R}^3$$

Projection in the plane: $\pi_{(x,y)}(\mathcal{C})$

Goal: Isolating singularities of $\pi_{(x,y)}(\mathcal{C})$

Motivations: Computing the topology of $\pi_{(x,y)}(\mathcal{C})$



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$$\mathcal{C} : \begin{cases} P(x,y,z) & = 0 \\ Q(x,y,z) & = 0 \end{cases}, (x,y,z) \in \mathbb{R}^3$$

Projection in the plane: $\pi_{(x,y)}(\mathcal{C})$

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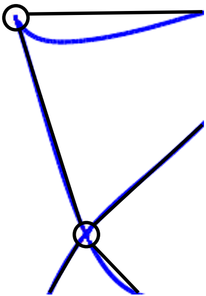


Isolating singularities

$$\mathcal{B} = \{(x, y) \in \mathbb{R}^2 | r(x, y) = 0\},$$

Singularities of $\mathcal{B}$ are the solutions of:

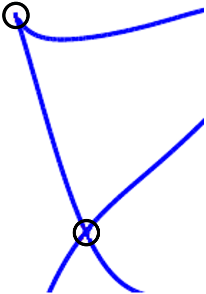
$$\begin{cases} r(x,y) = 0 \\ \frac{\partial r}{\partial x}(x,y) = 0 \\ \frac{\partial r}{\partial y}(x,y) = 0 \end{cases}$$

...that is over-determined

...that has solutions of multiplicity 2

when r is a polynomial:

Gröbner Basis, RUR



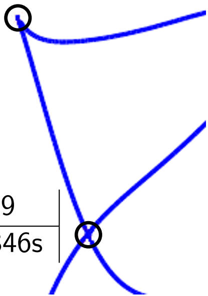
Isolating singularities of apparent contours: algebraic case

$\mathcal{B} = \{(x,y) \in \mathbb{R}^2 | r(x,y) = 0\}$, where $r(x,y) = Res(P, P_z, z)(x,y)$

Singularities of $\mathcal{B}$ are the solutions of:

$$\begin{cases} r(x,y) = 0 \\ \frac{\partial r}{\partial x}(x,y) = 0 \\ \frac{\partial r}{\partial y}(x,y) = 0 \end{cases}$$

P,	degree 6,	bit-size 8,	84 monomials
r,	degree 30,	bit-size 111,	496 monomials
$\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}$,	degree 29,	bit-size 115,	465 monomials



degree of P	5	6	7	8	9
time with RSCube*	3.1s	32s	254s	1898s	9346s

* F. Rouillier

Isolating singularities of apparent contours: algebraic case

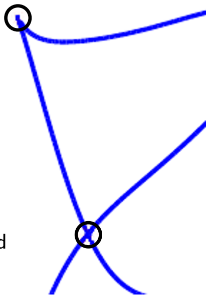
$\mathcal{B} = \{(x, y) \in \mathbb{R}^2 | r(x, y) = 0\}$, where $r(x, y) = Res(P, P_z, z)(x, y)$

Singularities of $\mathcal{B}$ are the regular solutions of:

$$(\mathcal{S}_2) \begin{cases} s_{10}(x, y) = 0 \\ s_{11}(x, y) = 0 \end{cases} \quad \text{s.t. } s_{22}(x, y) \neq 0$$

...where s_{10}, s_{11}, s_{22} are coefficients in the subresultant chain.

P,	degree 6,	bit-size 8,	84 monomials
r,	degree 30,	bit-size 111,	496 monomials
$\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}$,	degree 29,	bit-size 115,	465 monomials
s_{11}, s_{10},	degree 20,	bit-size 89,	231 monomials
s_{22},	degree 12,	bit-size 65,	91 monomials

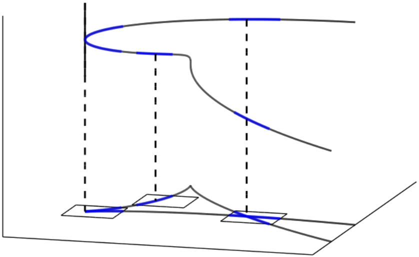


[IMP15] Rémi Imbach, Guillaume Moroz, and Marc Pouget.
Numeric certified algorithm for the topology of resultant and discriminant curves.
Research Report RR-8653, Inria, April 2015.

Contributions

P, Q generic analytic maps
$\rightarrow$ characterizing singularities of $\pi_{(x,y)}(\mathcal{C})$
$\rightarrow$ describing it as regular solutions of the ball system

P, Q generic polynomials
$\rightarrow$ numerical solving of the ball system
$\rightarrow$ certified numerical solving of the ball system



Assumptions on $\mathcal{C}$

(A_1) $\mathcal{C}$ is smooth

(A_2) $(\alpha, \beta) \in \mathbb{R}^2 \Rightarrow P(\alpha, \beta, z) = Q(\alpha, \beta, z) = 0$ has at most 2 real roots counted with multiplicities

(A_3) $\{(\alpha, \beta) \in \mathbb{R}^2 | P(\alpha, \beta, z) = Q(\alpha, \beta, z) = 0$ has two solutions counted with multiplicities $\}$ is a discrete set

(A_4) $\pi_{(x,y)}$ is a proper map (the pre-image of a compact is a compact)

Classification of Singularities of $\pi_{(x,y)}(\mathcal{C})$

(A_1) ...

(A_2) $(\alpha, \beta) \in \mathbb{R}^2 \Rightarrow P(\alpha, \beta, z) = Q(\alpha, \beta, z) = 0$ has at most 2 real roots counted with multiplicities

(A_3) ...

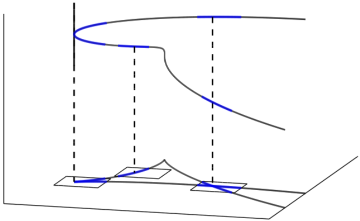
Let $(\alpha, \beta) \in \pi_{(x,y)}(\mathcal{C})$. $P(\alpha, \beta, z) = Q(\alpha, \beta, z) = 0$ has:

one root of multiplicity one

two roots of multiplicities one

one root of multiplicity two $\Leftrightarrow P_z(\alpha, \beta, z) = Q_z(\alpha, \beta, z) = 0$

Classification of Singularities of $\pi_{(x,y)}(\mathcal{C})$



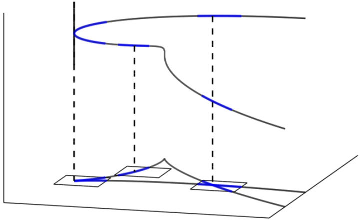
Let $(\alpha, \beta) \in \pi_{(x,y)}(\mathcal{C})$. (α, β) has:

one antecedent with a non vertical tangent

two antecedents

one antecedent with a vertical tangent

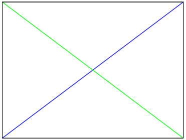
Classification of Singularities of $\pi_{(x,y)}(\mathcal{C})$



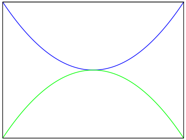
Let $(\alpha, \beta) \in \pi_{(x,y)}(\mathcal{C})$. If (α, β) has:

Lemma 1. one antecedent with a non vertical tangent then (α, β) is a regular point of $\pi_{(x,y)}(\mathcal{C})$

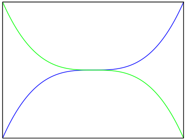
Classification of Singularities of $\pi_{(x,y)}(\mathcal{C})$



$k = 1$



$k = 2$



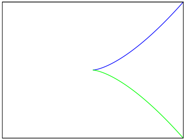
$k = 3$

Let $(\alpha, \beta) \in \pi_{(x,y)}(\mathcal{C})$. If (α, β) has:

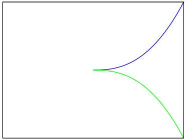
Lemma 1. one antecedent with a non vertical tangent
then (α, β) is a regular point of $\pi_{(x,y)}(\mathcal{C})$

Lemma 2. two antecedents
then $\pi_{(x,y)}(\mathcal{C})$ is locally diffeomorphic to $x^2 - y^{2k} \quad = 0$, with $k \geq 1$.

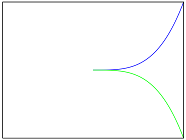
Classification of Singularities of $\pi_{(x,y)}(\mathcal{C})$



$k = 1$



$k = 2$



$k = 3$

Let $(\alpha, \beta) \in \pi_{(x,y)}(\mathcal{C})$. If (α, β) has:

Lemma 1. one antecedent with a non vertical tangent
then (α, β) is a regular point of $\pi_{(x,y)}(\mathcal{C})$

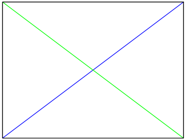
Lemma 2. two antecedents
then $\pi_{(x,y)}(\mathcal{C})$ is locally diffeomorphic to $x^2 - y^{2k} = 0$, with $k \geq 1$.

Lemma 3. one antecedent with a vertical tangent
then $\pi_{(x,y)}(\mathcal{C})$ is locally diffeomorphic to $x^2 + y^{2k+1} = 0$, with $k \geq 1$.

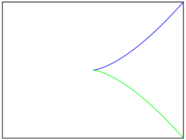
Classification of Singularities of $\pi_{(x,y)}(\mathcal{C})$

Assumptions on $\mathcal{C}$:

(A_5) Singularities of $\pi_{(x,y)}(\mathcal{C})$ are either nodes, or ordinary cusps



$k = 1$



$k = 1$

Lemma 2. two antecedents
then (α, β) is a node $(k = 1)$ of $\pi_{(x,y)}(\mathcal{C})$.

Lemma 3. one antecedent with a vertical tangent
then (α, β) is an ordinary cusp $(k = 1)$ of $\pi_{(x,y)}(\mathcal{C})$.

Assumptions on $\mathcal{C}$

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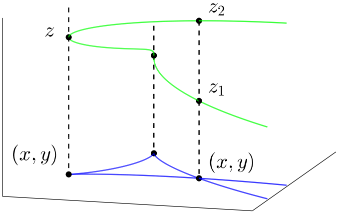
(A_4) $\pi_{(x,y)}$ is a proper map (the pre-image of a compact is a compact)

(A_5) Singularities of $\pi_{(x,y)}(\mathcal{C})$ are either nodes, or ordinary cusps

$(A_1), (A_2), (A_3), (A_5)$ hold for generic maps
(A_4) holds at least for generic polynomials

Isolating singularities:



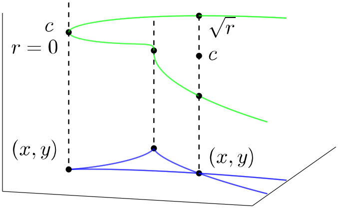
Lemma 2: (x, y) is a node of $\pi_{(x,y)}(\mathcal{C}) \Leftrightarrow (x, y, z_1, z_2)$ satisfies:

$$P(x, y, z_1) = Q(x, y, z_1) = P(x, y, z_2) = Q(x, y, z_2) = 0$$

Lemma 3: (x, y) is a cusp of $\pi_{(x,y)}(\mathcal{C}) \Leftrightarrow (x, y, z)$ satisfies:

$$P(x, y, z) = Q(x, y, z) = P_z(x, y, z) = Q_z(x, y, z) = 0$$

Isolating singularities: the Ball system

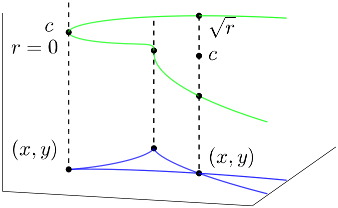


c: center of z_1, z_2
$r = \|cz_1\|_2^2$

Singularities of $\pi_{(x,y)}(\mathcal{C})$ are exactly the real solutions of:

$$(\mathcal{S}_4) \left\{ \begin{array}{rl} \frac{1}{2}(P(x,y,c+\sqrt{r}) + P(x,y,c-\sqrt{r})) & = 0 \\ \frac{1}{2\sqrt{r}}(P(x,y,c+\sqrt{r}) - P(x,y,c-\sqrt{r})) & = 0 \\ \frac{1}{2}(Q(x,y,c+\sqrt{r}) + Q(x,y,c-\sqrt{r})) & = 0 \\ \frac{1}{2\sqrt{r}}(Q(x,y,c+\sqrt{r}) - Q(x,y,c-\sqrt{r})) & = 0 \end{array} \right.$$

Isolating singularities: the Ball system



c: center of z_1, z_2
$r = \|cz_1\|_2^2$

Singularities of $\pi_{(x,y)}(\mathcal{C})$ are exactly the real solutions of:

when $r = 0$

$$(\mathcal{S}_4)\begin{cases} P(x,y,c) & = 0 \\ P_z(x,y,c) & = 0 \\ Q(x,y,c) & = 0 \\ Q_z(x,y,c) & = 0 \end{cases}$$

Isolating singularities: the Ball system

Lemma 5. Under the Assumptions $(A_1) - (A_4)$, all the solutions of $\mathcal{S}_4$ in $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^+$ are regular if and only if (A_5) is satisfied.

Lemma 4. Singularities of $\pi_{(x,y)}(\mathcal{C})$ are exactly the real solutions of:

$$(\mathcal{S}_4)\left\{\begin{array}{rl} \frac{1}{2}(P(x,y,c+\sqrt{r})+P(x,y,c-\sqrt{r})) & =0 \\ \frac{1}{2\sqrt{r}}(P(x,y,c+\sqrt{r})-P(x,y,c-\sqrt{r})) & =0 \\ \frac{1}{2}(Q(x,y,c+\sqrt{r})+Q(x,y,c-\sqrt{r})) & =0 \\ \frac{1}{2\sqrt{r}}(Q(x,y,c+\sqrt{r})-Q(x,y,c-\sqrt{r})) & =0 \end{array}\right.$$

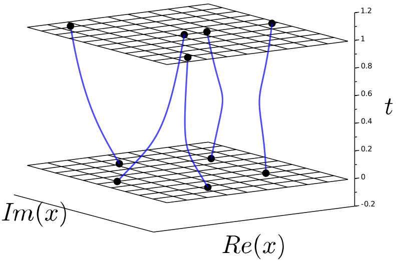
Numerical Solving

P, Q are polynomials $\Rightarrow$ equations of $\mathcal{S}_4$ are polynomials

Homotopy

- Solutions of $\mathcal{S}_4$ are approximated
- Solves $\mathcal{S}_4$ in $\mathbb{C}^4$ $\Rightarrow$ Singularities are isolated in $\mathbb{R}^2$
- Can be certified
  - for dense polynomials with Bézout bound of $\mathcal{S}_4$
  - with a certified path tracker

Example:
Target system:
$x^5 + 3x^2 + x = 0$
Initial system:
$\gamma\ (x^5 - 1) = 0,\ \gamma \in \mathbb{C}$
Homotopy function:
$H(x, t) = (1 - t)F_0(x) + tF(x)$



Numerical Solving

P, Q are polynomials $\Rightarrow$ equations of $\mathcal{S}_4$ are polynomials

Homotopy

- Solutions of $\mathcal{S}_4$ are approximated
- Solves $\mathcal{S}_4$ in $\mathbb{C}^4$ $\Rightarrow$ Singularities are isolated in $\mathbb{R}^2$
- Can be certified
- Implementations: `Bertini`[1] allows Adaptive Multi Precision (AMP)

[1] https://bertini.nd.edu/

Isolation of singularities of an apparent contour

Datas: Random dense polynomials of degree d, bit-size 8

Numerical results: Isolating singularities in $\mathbb{R}^2$

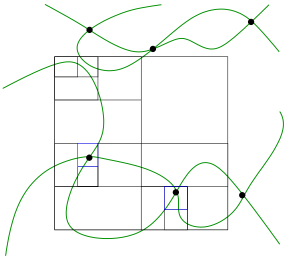
	sub-resultant system $\mathcal{S}_2$			ball system $\mathcal{S}_4$		
	DP		AMP	DP		AMP
d	t	Missed Sols	t	t	Missed Sols	t
5	3.638	0	147.852	3.818	2	15.01
6	54.49	1	1005	20.80	1	165.7
7	617.9	6	$\geq$ 3000	88.50	0	1147
8	2799	885	$\geq$ 3000	319.3	0	$\geq$ 3000
9	$\geq$ 3000	1178	$\geq$ 3000	935.6	2	$\geq$ 3000

means on 5 examples of sequential times in seconds on a Intel(R) Xeon(R) CPU L5640 @ 2.27GHz machine.

Certified Numerical Solving

Subdivision or Interval Solver or Branch and Bound approach

- Solutions of $\mathcal{S}_4$ are isolated in boxes
- Arbitrary arithmetic precision $\Rightarrow$ Termination $\Rightarrow$ Correction
- Solves $\mathcal{S}_4$ in $\mathbf{D} \subset \mathbb{R}^4$



Certified Numerical Solving

P, Q are polynomials $\Rightarrow$ equations of $\mathcal{S}_4$ are polynomials

Subdivision or Interval Solver or Branch and Bound approach

- Solutions of $\mathcal{S}_4$ are isolated in boxes
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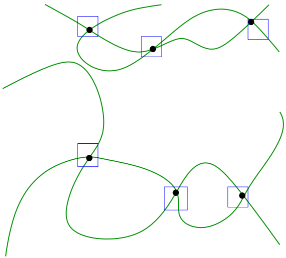
- Solves $\mathcal{S}_4$ in $\mathbf{D} \subseteq \mathbb{R}^4$ $\Rightarrow$ Singularities are isolated in $\mathbf{B} \subseteq \mathbb{R}^2$



[Sta95] Volker Stahl.
Interval Methods for Bounding the Range of Polynomials and Solving Systems of Nonlinear Equations.
PhD thesis, Johannes Kepler University, Linz, Austria, 1995.

Certified Numerical Solving

Subdivision or Interval Solver or Branch and Bound approach

- Solutions of $\mathcal{S}_4$ are isolated in boxes
- Arbitrary arithmetic precision $\Rightarrow$ Termination $\Rightarrow$ Correction
- Solves $\mathcal{S}_4$ in $\mathbf{D} \subseteq \mathbb{R}^4 \Rightarrow$ Singularities are isolated in $\mathbf{B} \subseteq \mathbb{R}^2$
- Implementation: home made in C++
  - evaluation of polynomials with horner scheme $\rightarrow$ quick
  - evaluation of polynomials at order 2 $\rightarrow$ sharp

Isolation of singularities of an apparent contour

Datas: Random dense polynomials of degree d, bit-size 8

Numerical results: Isolating singularities in $\mathbb{R}^2$

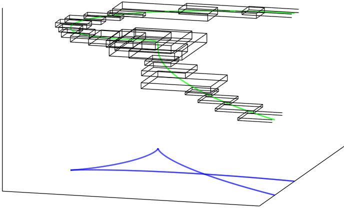
	sub-resultant system $\mathcal{S}_2$			ball system $\mathcal{S}_4$		
d	DP	AMP	Subdivision	DP	AMP	Subdivision
5	3.638	147.852	0.251	3.818	15.01	25.34
6	54.49	1005	1.353	20.80	165.7	11.38
7	617.9	$\geq$ 3000	124.1	88.50	1147	54.21
8	2799	$\geq$ 3000	57.72	319.3	$\geq$ 3000	99.22
9	$\geq$ 3000	$\geq$ 3000	54.74	935.6	$\geq$ 3000	95.11

means on 5 examples of sequential times in seconds on a Intel(R) Xeon(R) CPU L5640 @ 2.27GHz machine.

Enclosing $\mathcal{C}$ to restrain the solving domain of $\mathcal{S}_4$

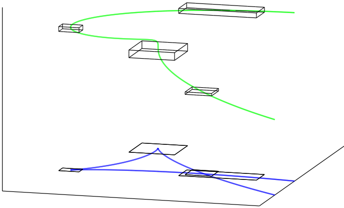
Enclosing $\mathcal{C}$ in a sequence of boxes:

- Certified path tracker
- 1 point on each C.C.: subdivision solver



Geometric characterization of nodes and cusps:

- 4D square system
- Subdivision Solver
- Restriction of the solving domain



Isolation of singularities of an apparent contour

Datas: Random dense polynomials of degree d, bit-size 8

Numerical results: Isolating singularities in $[-1, 1] \times [-1, 1]$

	sub-resultant system $\mathcal{S}_2$	ball system $\mathcal{S}_4$	
	Subdivision	Subdivision	Curve tracking & subdivision
d	t	t	t
5	0.05	24.8	1.25
6	0.50	8.40	2.36
7	4.44	43.8	4.13
8	37.9	70.2	5.91
9	23.1	45.6	5.30

means on 5 examples of sequential times in seconds on a Intel(R) Xeon(R) CPU L5640 @ 2.27GHz machine.

Questions ?