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Practical Advances in Complex Root Clustering

Collaborative and ongoing works

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Example

System: Let $\sigma \geq 3$ and $\boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{0}$ be:

$$\begin{cases} (z_1 - 2^{-\sigma})\ (z_1 + 2^{-\sigma}) & = & 0 \\ (z_2 + 2^{\sigma} z_1^2)\ (z_2 - 1) z_2 & = & 0 \end{cases}$$

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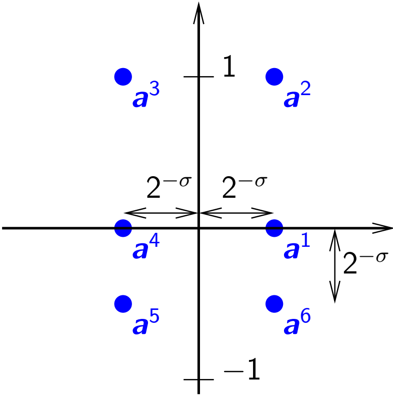

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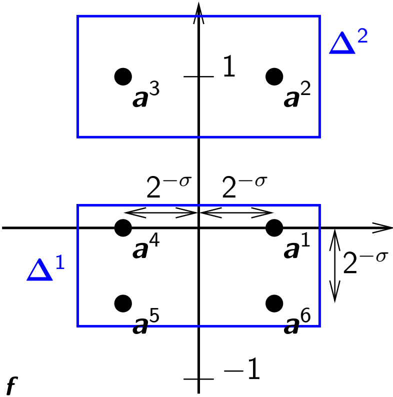

Natural clusters:

$(\boldsymbol{\Delta}^1, 4)$

$(\boldsymbol{\Delta}^2, 2)$

Notations: $m(\boldsymbol{a}, \boldsymbol{f})$: multiplicity of $\boldsymbol{a}$ as a sol. of $\boldsymbol{f}$

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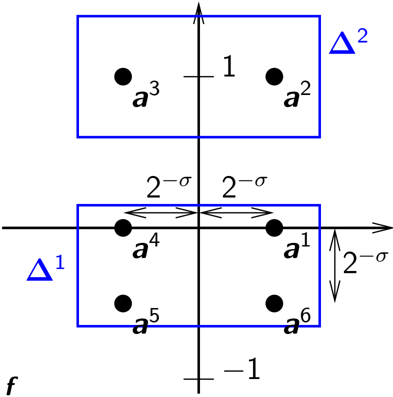

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Local solution Clustering Problem (LCP)

Input: a polynomial map $\boldsymbol{f} : \mathbb{C}^n \to \mathbb{C}^n$ (assume $\boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{0}$ is 0-dim),
a polybox $\boldsymbol{B} \subset \mathbb{C}^n$, the Region of Interest (RoI),
$\epsilon > 0$

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Definition: a pair $(\boldsymbol{\Delta}, m)$ is called natural cluster (relative to $\boldsymbol{f}$)
when it satisfies:

$$m = \#(\boldsymbol{\Delta}, \boldsymbol{f}) = \#(3\boldsymbol{\Delta}, \boldsymbol{f}) \geq 1$$

if $r(\boldsymbol{\Delta}) \leq \epsilon$, it is a natural ϵ-cluster

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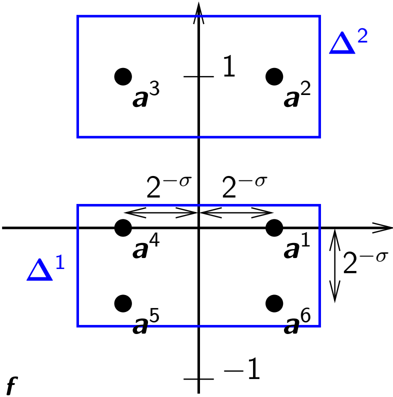

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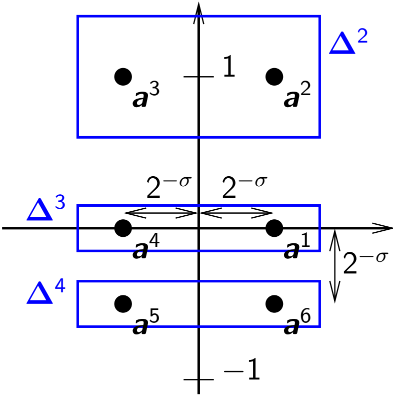

$(\boldsymbol{\Delta}^3, 3)$, $(\boldsymbol{\Delta}^4, 6)$ are not natural clusters

Why root clustering instead of root isolation?

Root isolation:

- input polynomials with $\mathbb{Z}$ or $\mathbb{Q}$ coefficients, or
- input polynomials squarefree

Root clustering:

- input polynomials with any $\mathbb{C}$ coefficients
- robust to multiple roots

Menu

0 - Univariate case:

[BSS+16] Ruben Becker, Michael Sagraloff, Vikram Sharma, Juan Xu, and Chee Yap.
Complexity analysis of root clustering for a complex polynomial.
In *ISSAC 16*, pages 71–78. ACM, 2016.

Near optimal: bit complexity $\widetilde{O}(d^2(\sigma + d))$
for the benchmark problem

Efficient implementation `Ccluster` described in

[IPY18] Rémi Imbach, Victor Y. Pan, and Chee Yap.
Implementation of a near-optimal complex root clustering algorithm.
In *Mathematical Software – ICMS 2018*, pages 235–244, Cham, 2018.

Notations: d, σ: degree, bit-size of f

Menu

0 - Univariate case:

1 - Multivariate triangular case

[IPY19] Rémi Imbach, Marc Pouget, and Chee Yap.
Clustering complex zeros of triangular systems of polynomials.
In *CASC 19, to appear in MCS*, 2019.

$$\begin{cases} f_1(z_1) & = & 0 \\ f_2(z_1, z_2) & = & 0 \\ \dots & & \\ f_n(z_1, z_2, \dots, z_n) & = & 0 \end{cases}, \deg_{z_i}(f_i) \geq 1$$

with: finite number of sols

Symbolic-Numeric solving of systems of polynomials:

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system	symbolic	numeric $\mathbb{R}$	symbolic	numeric $\mathbb{C}$	
$\mathcal{S}_4$	3.8	3.7	0.6	0.18	
$\mathcal{S}_5$	24.2	>1000	42.9	0.57	

seq. times in s on a `Intel(R) Core(TM) i7-7600U CPU @ 2.80GHz`
asked precision: 53 bits

$$\mathcal{S}_4 \left\{ \begin{array}{ll} z1^4 - 57 * z1^2 * z2 - 86 * z1 * z2^2 - 160 * z2^3 + 95 * z2^2 * z3 + 35 * z1^2 - 106 * z3 & = 0 \\ z2^4 - 64 * z2^3 - 190 * z1 * z2 + 186 * z1 * z3 - 119 * z2 * z3 + 188 * z3 + 93 & = 0 \\ z3^4 + 116 * z1 * z2^2 - 168 * z1 * z2 * z3 + 135 * z1 * z3^2 + 29 * z3^3 - 8 * z1 * z3 + 119 * z2 * z3 & = 0 \end{array} \right.$$

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Menu

0 - Univariate case:

1 - Multivariate triangular case

2 - Back to univariate case

- polynomials with real coefficients
- new counting test

[IP19] Rémi Imbach and Victor Y. Pan.
New practical advances in polynomial root clustering.
In *MACIS 19*, 2019.

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Oracle numbers and polynomials

Let $\alpha \in \mathbb{C}$.

Oracle for α: function $\mathcal{O}_\alpha : \mathbb{Z} \to \square\mathbb{C}$
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$\simeq$ oracles for the coeffs of f

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$\square\mathbb{C}[z_1, \ldots, z_n]$: polynomials with coefficients in $\square\mathbb{C}$

Outline of [BSS+16]

Counting test: $T^* : (\Delta, \mathcal{O}_f) \mapsto m \in \{-1, 0, \dots, d\}$
$T^*(\Delta, \mathcal{O}_f) \geq 0 \Rightarrow \#(\Delta, f) = m$

Discarding test: $T^0 : (\Delta, \mathcal{O}_f) \mapsto m \in \{-1, 0\}$
$T^0(\Delta, \mathcal{O}_f) = 0 \Rightarrow \#(\Delta, f) = 0$

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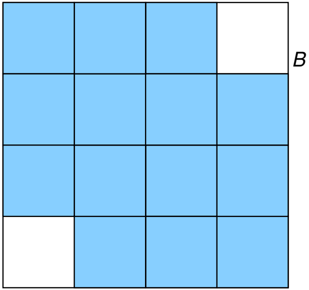

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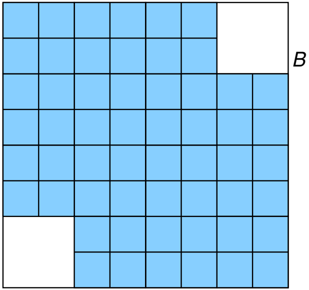

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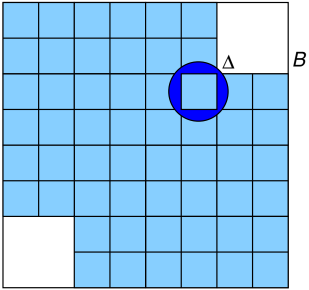

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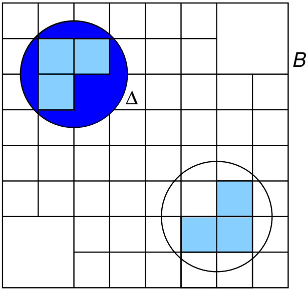
$T^*(\Delta, \mathcal{O}_f) \geq 0 \Rightarrow \#(\Delta, f) = m$

Discarding test: $T^0 : (\Delta, \mathcal{O}_f) \mapsto m \in \{-1, 0\}$
$T^0(\Delta, \mathcal{O}_f) = 0 \Rightarrow \#(\Delta, f) = 0$

Subdivision approach:



Notations: $\#(S, f)$: sum of multiplicities of roots of f in S
d: degree of f

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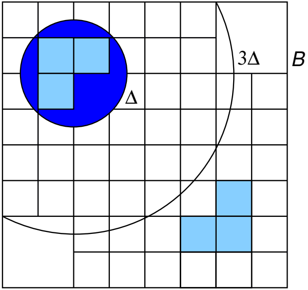

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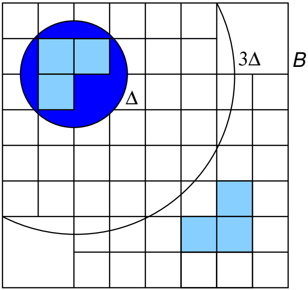

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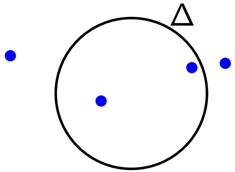
The Pellet's test

Pellet's Theorem: Let Δ be a complex disc centered in c and radius r.
Let $f \in \mathbb{C}[z]$, $d = \deg(f)$ and $f_\Delta = f(c + rz)$.

If $\exists\ 0 \leq m \leq d$ s.t.

$$|(f_\Delta)_m| > \sum_{i \neq k} |(f_\Delta)_i| \quad (1)$$

then f has exactly m roots in Δ.



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If f has no root in this annulus $\rightarrow$
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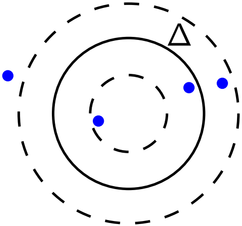

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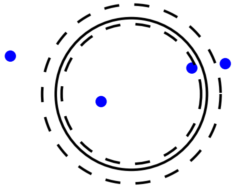

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PelletTest(Δ, f) *//Output in $\{-1, 0, 1, \ldots, d\}$*

1. compute f_Δ
2. **for** m **from** 0 **to** d **do**
3. **if** $|(f_\Delta)_m| > \sum_{i \neq k} |(f_\Delta)_i|$
4. **return** m *//m roots (with mult.) in Δ*
5. **return** -1 *//Roots near the boundary of Δ*

The soft Pellet's test: for interval polynomials

Pellet's Theorem: Let Δ be a complex disc centered in c and radius r.
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SoftCompare($\square a$, $\square b$) *//$\square a$, $\square b$ are real intervals*

Input: $\square a$, $\square b$ real intervals

Output: a number in $\{-2, -1, 1\}$ s.t.:

$1 \Rightarrow \square a > \square b$

$-1 \Rightarrow \square a < \square b$ or $\square a$, $\square b$ are too close

$-2 \Rightarrow \square a \cap \square b \neq \emptyset$

The soft Pellet's test: for interval polynomials

SoftPelletTest($\Delta, \square f$) *//Output in $\{-2, -1, 0, 1, \ldots, d\}$*

1. compute $\square f_\Delta$
2. **for** m **from** 0 **to** deg **do**
3. $R \leftarrow$ SoftCompare$(|(\square f_\Delta)_m|, \sum_{i \neq k} |(\square f_\Delta)_i|)$
4. **if** $R \geq 0$ **then return** m *//any $f \in \square f$ has m roots* // (with mult.) in Δ
5. **if** $R = -2$ **then return** -2 *//$\square f$ is too wide*
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Loop on precision:

$T^*(\Delta, \mathcal{O}_f)$ *//Output in $\{-1, 0, 1, \ldots, d\}$*

1. $L \leftarrow 53$, $\square f \leftarrow \mathcal{O}_f(L)$, $m \leftarrow$SoftPelletTest$(\Delta, \square f)$
2. **while** $m = -2$ **do**
3. $\quad L \leftarrow 2L$, $\square f \leftarrow \mathcal{O}_f(L)$, $m \leftarrow$SoftPelletTest$(\Delta, \square f)$
4. **return** m

Univariate root clustering algorithms

ClusterOracle: solves the LCP in 1D ([BSS+16])
T^* embedded in a subdivision framework
accepts oracle polynomials in input

[BSS+16] Ruben Becker, Michael Sagraloff, Vikram Sharma, Juan Xu, and Chee Yap.
Complexity analysis of root clustering for a complex polynomial.
In *ISSAC 16*, pages 71–78. ACM, 2016.

Univariate root clustering algorithms

ClusterOracle: solves the LCP in 1D ([BSS+16])
T^* embedded in a subdivision framework
accepts oracle polynomials in input

ClusterInterval: solves the LCP in 1D
Input: interval polynomial
Output: a flag in {**success**,**fail**}, a list of natural clusters
SoftPelletTest embedded in a subdivision framework
returns **fail** when SoftPelletTest returns -2

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Menu

0 - Univariate case:

1 - Multivariate triangular case

[IPY19] Rémi Imbach, Marc Pouget, and Chee Yap.
Clustering complex zeros of triangular systems of polynomials.
In *CASC 19, to appear in MCS*, 2019.

Rational, bivariate

$$\begin{cases} f_1(z_1) & = & 0 \\ f_2(z_1, z_2) & = & 0 \end{cases}, \deg_{z_i}(f_i) \geq 1, f_i \in \mathbb{Q}[z_1, z_2]$$

Oracle numbers and polynomials

Let $\alpha \in \mathbb{C}$.

Oracle for α: function $\mathcal{O}_\alpha : \mathbb{Z} \to \square\mathbb{C}$
s.t. $\alpha \in \mathcal{O}_\alpha(L)$ and $w(\mathcal{O}_\alpha(L)) \leq 2^{-L}$

Let $f \in \mathbb{C}[z_1, \ldots, z_n]$

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$\simeq$ oracles for the coeffs of f

Let $f_2 \in \mathbb{Q}[z_1, z_2]$ and $\alpha_1 \in \mathbb{C}$

Partial specialization of f_2: $f_2(\alpha_1) \in \mathbb{C}[z_2]$

Notations: $\square\mathbb{C}$: set of complex interval
$\square\mathbb{C}[z_1, \ldots, z_n]$: polynomials with coefficients in $\square\mathbb{C}$

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Number of solutions in a polydisc

Let $\boldsymbol{\Delta} = (\Delta_1, \Delta_2)$ and $\boldsymbol{m} = (m_1, m_2)$.

Proposition 1: Suppose

(*i*) f_1 has m_1 roots in Δ_1 with multiplicity

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Proof: direct consequence of

Theorem [ZFX11]: Let $\boldsymbol{\alpha} \in Z(\mathbb{C}^2, \boldsymbol{f})$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$. Then

$$m(\boldsymbol{\alpha}, \boldsymbol{f}) = m(\alpha_2, f_2(\alpha_1)) \times m(\alpha_1, f_1)$$

[ZFX11] Zhihai Zhang, Tian Fang, and Bican Xia.
Real solution isolation with multiplicity of zero-dimensional triangular systems.
Science China Information Sciences, 54(1):60–69, 2011.

Example

System: Let $\sigma \geq 3$ and $\boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{0}$ be:

$$\begin{cases} (z_1 - 2^{-\sigma})^2 (z_1 + 2^{-\sigma}) & = & 0 \\ (z_2 + 2^{\sigma} z_1^2)^2 (z_2 - 1) z_2 & = & 0 \end{cases}$$

Solutions: $\boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{0}$ has 6 solutions, all real:

$$\begin{array}{llll}
\boldsymbol{a}^1 = (2^{-\sigma} & , & 0) & \leftarrow m(\boldsymbol{a}^1, \boldsymbol{f}) = 2 = 1 \times 2 \\
\boldsymbol{a}^2 = (2^{-\sigma} & , & 1) & \leftarrow m(\boldsymbol{a}^2, \boldsymbol{f}) = 2 = 1 \times 2 \\
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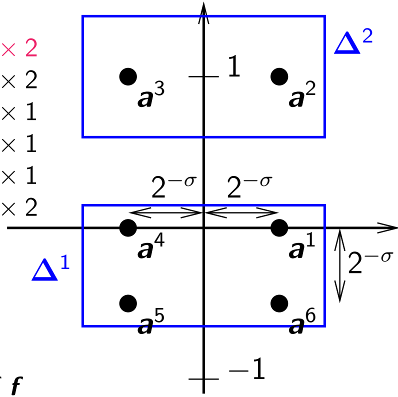

Natural clusters:

$(\boldsymbol{\Delta}^1, 9)$

$(\boldsymbol{\Delta}^2, 3)$

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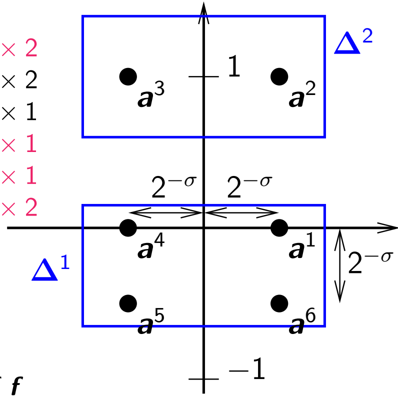

Natural clusters:

$(\boldsymbol{\Delta}^1, 9) \leftarrow 9 = 3 \times 3$

$(\boldsymbol{\Delta}^2, 3) \leftarrow 3 = 1 \times 3$

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Then $\boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{0}$ has $m_2 \times m_1$ solutions in $\boldsymbol{\Delta}$ with multiplicity.

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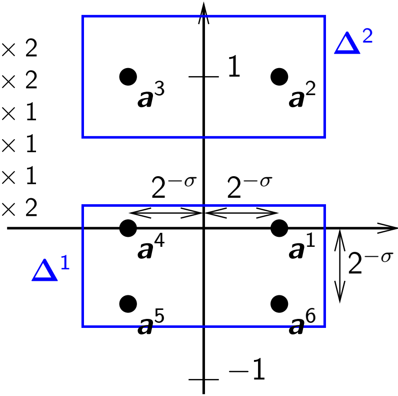

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Natural towers:

$(\Delta^1, (3,3))$
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(i) (Δ_1, m_1) is a natural cluster relative to f_1

(ii) $\forall \alpha_1 \in \Delta_1$, (Δ_2, m_2) is a natural cluster relative to $f_2(\alpha_1)$

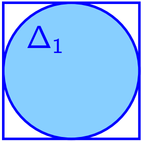
$\boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{0}$ has $m_2 \times m_1$ solutions in $\boldsymbol{\Delta}$ with multiplicity.

Proposition 3: Suppose

(i) SoftPelletTest(Δ_1, f_1) returns $m_1 \geq 1$

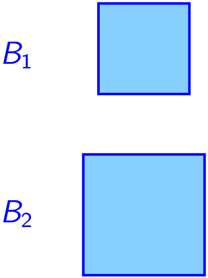
(ii) SoftPelletTest$(\Delta_2, f_2(\square\Delta_1))$ returns $m_2 \geq 1$

Then $(\boldsymbol{\Delta}, \boldsymbol{m})$ is a natural tower relative to $\boldsymbol{f}$.



$\square\Delta_1$

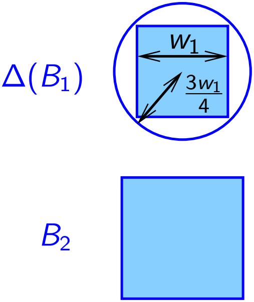
Main data structure



A **tower** is a triple $\mathcal{T} = \langle \ell, \boldsymbol{B}, \boldsymbol{L} \rangle$ where

- ℓ is an integer in $\{0, 1, 2\}$ called **level**
- $\boldsymbol{B} = (B_1, B_2)$ is a polybox called **domain**
- $\boldsymbol{L} = (L_1, L_2)$ is a vector in $(\mathbb{Z})^2$ called **precision**

Main data structure



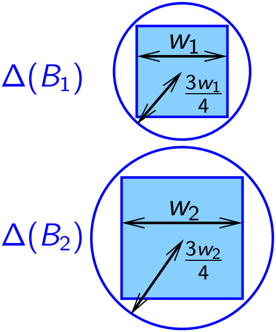
A **tower** is a triple $\mathcal{T} = \langle \ell, \boldsymbol{B}, \boldsymbol{L} \rangle$ where

- ℓ is an integer in $\{0, 1, 2\}$ called **level**
- $\boldsymbol{B} = (B_1, B_2)$ is a polybox called **domain**
- $\boldsymbol{L} = (L_1, L_2)$ is a vector in $(\mathbb{Z})^2$ called **precision**

We will garantee that if ℓ =1, $\exists$ m_1 so that:

(i) SoftPelletTest$(\Delta(B_1), f_1)$ returns m_1 and $r(\Delta(B_1)) < 2^{-L_1}$

Main data structure



A **tower** is a triple $\mathcal{T} = \langle \ell, \boldsymbol{B}, \boldsymbol{L} \rangle$ where

- ℓ is an integer in $\{0, 1, 2\}$ called **level**
- $\boldsymbol{B} = (B_1, B_2)$ is a polybox called **domain**
- $\boldsymbol{L} = (L_1, L_2)$ is a vector in $(\mathbb{Z})^2$ called **precision**

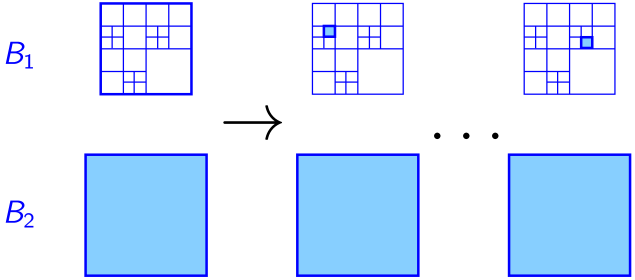
We will garantee that if $\ell = 2$, $\exists(m_1, m_2)$ so that:

(i) SoftPelletTest($\Delta(B_1), f_1$) returns m_1 and $r(\Delta(B_1)) < 2^{-L_1}$

(ii) SoftPelletTest($\Delta(B_2), f_2(\square\Delta(B_1))$) returns m_2 and $r(\Delta(B_2)) < 2^{-L_2}$

From proposition 3: $(\Delta(\boldsymbol{B}), \boldsymbol{m})$ is a natural tower (relative to $\boldsymbol{f}$) and $\boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{0}$ has $m_2 \times m_1$ sols in $\Delta(\boldsymbol{B})$ with mult.

Lift of a tower from level 0 to level 1

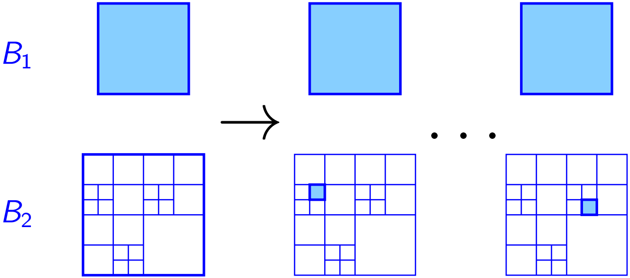


Cluster1($\boldsymbol{f}, \mathcal{T}$) *//for f with exact coefficients*

Input: $\boldsymbol{f} = (f_1, f_2)$, $\mathcal{T} = \langle \ell, \boldsymbol{B}, \boldsymbol{L} \rangle$ a tower at any level
Output: a list of towers at level 1

1. calls ClusterOracle ([BSS+16]) for f_1, B_1, 2^{-L_1}

Lift of a tower from level 1 to level 2



Cluster2($\boldsymbol{f}, \mathcal{T}$) *//for f with exact coefficients*

Input: $\boldsymbol{f} = (f_1, f_2)$, $\mathcal{T} = \langle \ell, \boldsymbol{B}, \boldsymbol{L} \rangle$ a tower at level 1
Output: a flag in {**success**, **fail**} and a list of towers at level 2

1. calls ClusterInterval for $f_2(\square\Delta(B_1))$, B_2, 2^{-L_2}
fail if SoftPelletTest returns -2 (*i.e.* not enough prec. on $\square\Delta(B_1)$)

Main algorithm

ClusterTri($\boldsymbol{f}, \boldsymbol{B}, L$) *//for f with exact coefficients*

Input: a triangular system $\boldsymbol{f}(\boldsymbol{z}) = \mathbf{0}$, a polybox $\boldsymbol{B}$, $L > 0$
Output: a set of natural 2^{-L}-towers solving the LCP

1. $Q.push(\langle 0, \boldsymbol{B}, (L, L)\rangle)$
2. **while** Q contains towers of level < 2 **do**
3. $\mathcal{T} = \langle \ell, \boldsymbol{B}, (L_1, L_2)\rangle \leftarrow Q.pop()$ with $\ell < 2$
4.
5.
6.
7.
8.
9.
10.
11.
12. **return** Q

Main algorithm

ClusterTri($\boldsymbol{f}, \boldsymbol{B}, L$) *//for f with exact coefficients*

Input: a triangular system $\boldsymbol{f}(\boldsymbol{z}) = \mathbf{0}$, a polybox $\boldsymbol{B}$, $L > 0$
Output: a set of natural 2^{-L}-towers solving the LCP

```
Q.push(⟨0, B, (L, L)⟩)
while Q contains towers of level < 2 do
        T = ⟨ℓ, B, (L_1, L_2)⟩ ← Q.pop() with ℓ < 2
        if ℓ = 0 then
                Q.push(Cluster1( f, T ))
        else
return Q
```

Main algorithm

ClusterTri($\boldsymbol{f}, \boldsymbol{B}, L$) *//for f with exact coefficients*

Input: a triangular system $\boldsymbol{f}(\boldsymbol{z}) = \mathbf{0}$, a polybox $\boldsymbol{B}$, $L > 0$
Output: a set of natural 2^{-L}-towers solving the LCP

$Q.push(\langle 0, \boldsymbol{B}, (L, L)\rangle)$
while Q contains towers of level < 2 **do**
  $\mathcal{T} = \langle \ell, \boldsymbol{B}, (L_1, L_2)\rangle \leftarrow Q.pop()$ with $\ell < 2$
  if $\ell = 0$ **then**
    $Q.push($Cluster1$(\boldsymbol{f}, \mathcal{T}))$
  else
    flag, S$\leftarrow$Cluster2$(\boldsymbol{f}, \mathcal{T})$
    if flag $=$ **success then**
      $Q.push(S)$
    else *// not enough precision on B_1*
      $Q.push(\langle 0, \boldsymbol{B}, (2L_1, L_2)\rangle)$
return Q

Our implementation

`Ccluster`: library in `C` based on

- FLINT[1]: arithmetic for the geometric algorithm
- Arb[2]: arbitrary precision floating arithmetic with error bounds

Available at `https://github.com/rimbach/Ccluster`

`Ccluster.jl`: package for julia[3] based on $\mathbb{N}e^m\mathcal{O}$[4]

- interface for `Ccluster`
- `Tcluster`: implemetation of ClusterTri

Available at `https://github.com/rimbach/Ccluster.jl`

[1] `https://github.com/wbhart/flint2`
[2] `http://arblib.org/`
[3] `https://julialang.org/`
[4] `http://nemocas.org/`

Benchmark: systems

Type of a triangular system:
$\boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{0}$ has type $(d_1, \ldots, d_n)$ if f_i has degree d_i in z_i, $\forall 1 \leq i \leq n$

Table: for each type, average on 5 random dense systems
seq. times on a `Intel(R) Core(TM) i7-7600U CPU @ 2.80GHz`

type								
Systems with only simple solutions								
(9,9,9)								
(6,6,6,6)								
(9,9,9,9)								
(6,6,6,6,6)								
(9,9,9,9,9)								
(2,2,2,2,2,2,2,2,2,2)								
Systems with multiple solutions								
(9,9)								
(6,6,6)								
(9,9,9)								
(6,6,6,6)								

Benchmark: local vs global comparison

Type of a triangular system:
$\boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{0}$ has type $(d_1, \ldots, d_n)$ if f_i has degree d_i in z_i, $\forall 1 \leq i \leq n$

Table: for each type, average on 5 random dense systems
seq. times on a `Intel(R) Core(TM) i7-7600U CPU @ 2.80GHz`

	Tcluster **local**		Tcluster **global**					
type	(#Clus, #Sols)	t (s)	(#Clus, #Sols)	t (s)				
Systems with only simple solutions								
(9,9,9)	(149 : 149)	0.24	(729 : 729)	1.21				
(6,6,6,6)	(63.4 : 63.4)	0.10	(1296 : 1296)	1.73				
(9,9,9,9)	(559 : 559)	1.06	(6561 : 6561)	12.9				
(6,6,6,6,6)	(155 : 155)	0.37	(7776 : 7776)	11.1				
(9,9,9,9,9)	(1739 : 1739)	4.83	(59049 : 59049)	113				
(2,2,2,2,2,2,2,2,2,2)	(0 : 0)	0.13	(1024 : 1024)	2.42				
Systems with multiple solutions								
(9,9)	(23.8: 13.6)	0.03	(81 : 45)	0.15				
(6,6,6)	(35.2: 8.80)	0.05	(216 : 54)	0.24				
(9,9,9)	(113 : 37.6)	0.22	(729 : 225)	1.06				
(6,6,6,6)	(81.6: 10.2)	0.21	(1296: 162)	1.28				

Tcluster **local** : $\boldsymbol{B} = ([-1,1] + \imath[-1,1])^2$, $\epsilon = 2^{-53}$
Tcluster **global**: $\boldsymbol{B}$ chosen with upper bound for roots

Benchmark: extern comparison

Type of a triangular system:
$\boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{0}$ has type $(d_1, \ldots, d_n)$ if f_i has degree d_i in z_i, $\forall 1 \leq i \leq n$

Table: for each type, average on 5 random dense systems
seq. times on a `Intel(R) Core(TM) i7-7600U CPU @ 2.80GHz`

	`Tcluster` **local**		`Tcluster` **global**		`HomCont.jl`			
type	(`#Clus`, `#Sols`)	t (s)	(`#Clus`, `#Sols`)	t (s)	`#Sols`	t (s)		
Systems with only simple solutions								
(9,9,9)	(149 : 149)	0.24	(729 : 729)	1.21	729	4.21		
(6,6,6,6)	(63.4 : 63.4)	0.10	(1296 : 1296)	1.73	1296	4.70		
(9,9,9,9)	(559 : 559)	1.06	(6561 : 6561)	12.9	6561	14.0		
(6,6,6,6,6)	(155 : 155)	0.37	(7776 : 7776)	11.1	7776	11.5		
(9,9,9,9,9)	(1739 : 1739)	4.83	(59049 : 59049)	113	59049	116		
(2,2,2,2,2,2,2,2,2,2)	(0 : 0)	0.13	(1024 : 1024)	2.42	1024	4.84		
Systems with multiple solutions								
(9,9)	(23.8: 13.6)	0.03	(81 : 45)	0.15	33.6	3.27		
(6,6,6)	(35.2: 8.80)	0.05	(216 : 54)	0.24	53.2	2.75		
(9,9,9)	(113 : 37.6)	0.22	(729 : 225)	1.06	159	28.4		
(6,6,6,6)	(81.6: 10.2)	0.21	(1296: 162)	1.28	134	8.06		

`Tcluster` **local** : $\boldsymbol{B} = ([-1,1] + \imath[-1,1])^2$, $\epsilon = 2^{-53}$
`Tcluster` **global**: $\boldsymbol{B}$ chosen with upper bound for roots
`HomCont.jl`: `HomotopyContinuation.jl`

Benchmark:

Type of a triangular system:

$\boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{0}$ has type $(d_1, \ldots, d_n)$ if f_i has degree d_i in z_i, $\forall 1 \leq i \leq n$

Table: for each type, average on 5 random dense systems

seq. times on a `Intel(R) Core(TM) i7-7600U CPU @ 2.80GHz`

	`Tcluster` **local**		`Tcluster` **global**		`HomCont.jl`			
type	(`#Clus`, `#Sols`)	t (s)	(`#Clus`, `#Sols`)	t (s)	`#Sols`	t (s)		
Systems with only simple solutions								
(9,9,9)	(149 : 149)	0.24	(729 : 729)	1.21	729	4.21		
(6,6,6,6)	(63.4 : 63.4)	0.10	(1296 : 1296)	1.73	1296	4.70		
(9,9,9,9)	(559 : 559)	1.06	(6561 : 6561)	12.9	6561	14.0		
(6,6,6,6,6)	(155 : 155)	0.37	(7776 : 7776)	11.1	7776	11.5		
(9,9,9,9,9)	(1739 : 1739)	4.83	(59049 : 59049)	113	59049	116		
(2,2,2,2,2,2,2,2,2,2)	(0 : 0)	0.13	(1024 : 1024)	2.42	1024	4.84		
Systems with multiple solutions								
(9,9)	(23.8: 13.6)	0.03	(81 : 45)	0.15	33.6	3.27		
(6,6,6)	(35.2: 8.80)	0.05	(216 : 54)	0.24	53.2	2.75		
(9,9,9)	(113 : 37.6)	0.22	(729 : 225)	1.06	159	28.4		
(6,6,6,6)	(81.6: 10.2)	0.21	(1296: 162)	1.28	134	8.06		

`Tcluster` **local** : $\boldsymbol{B} = ([-1,1] + \imath[-1,1])^2$, $\epsilon = 2^{-53}$

`Tcluster` **global**: $\boldsymbol{B}$ chosen with upper bound for roots

`HomCont.jl`: `HomotopyContinuation.jl`

Benchmark:

Type of a triangular system:
$\boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{0}$ has type $(d_1, \ldots, d_n)$ if f_i has degree d_i in z_i, $\forall 1 \leq i \leq n$

Table: for each type, average on 5 random dense systems
seq. times on a `Intel(R) Core(TM) i7-7600U CPU @ 2.80GHz`

	Tcluster **local**		Tcluster **global**		HomCont.jl		triang_solve	
type	(#Clus, #Sols)	t (s)	(#Clus, #Sols)	t (s)	#Sols	t (s)	#Sols	t (s)
Systems with only simple solutions								
(9,9,9)	(149 : 149)	0.24	(729 : 729)	1.21	729	4.21	729	0.37
(6,6,6,6)	(63.4 : 63.4)	0.10	(1296 : 1296)	1.73	1296	4.70	1296	0.93
(9,9,9,9)	(559 : 559)	1.06	(6561 : 6561)	12.9	6561	14.0	6561	8.57
(6,6,6,6,6)	(155 : 155)	0.37	(7776 : 7776)	11.1	7776	11.5	7776	19.1
(9,9,9,9,9)	(1739 : 1739)	4.83	(59049 : 59049)	113	59049	116	59049	702
(2,2,2,2,2,2,2,2,2,2)	(0 : 0)	0.13	(1024 : 1024)	2.42	1024	4.84	1024	3.9
Systems with multiple solutions								
(9,9)	(23.8: 13.6)	0.03	(81 : 45)	0.15	33.6	3.27	45	0.03
(6,6,6)	(35.2: 8.80)	0.05	(216 : 54)	0.24	53.2	2.75	54	0.05
(9,9,9)	(113 : 37.6)	0.22	(729 : 225)	1.06	159	28.4	225	0.23
(6,6,6,6)	(81.6: 10.2)	0.21	(1296: 162)	1.28	134	8.06	162	0.15

`Tcluster` **local** : $\boldsymbol{B} = ([-1,1] + \imath[-1,1])^2$, $\epsilon = 2^{-53}$
`Tcluster` **global**: $\boldsymbol{B}$ chosen with upper bound for roots
`HomCont.jl`: `HomotopyContinuation.jl`
`triang_solve`: Singular solver for triangular systems

Menu

0 - Univariate case:

1 - Multivariate triangular case

2 - Back to univariate case

- polynomials with real coefficients
- new counting test

[IP19] Rémi Imbach and Victor Y. Pan.
New practical advances in polynomial root clustering.
In *MACIS 19*, 2019.

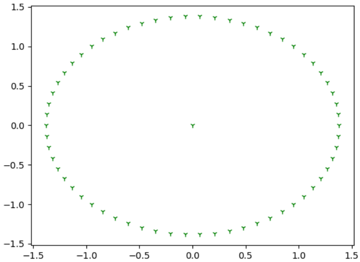
Pols with real coefficients

Example:

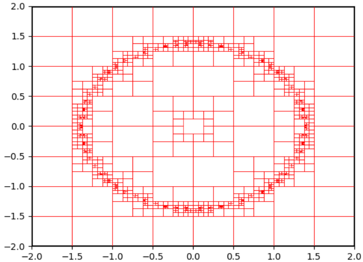
$$\mathtt{Mign}_d(z) = z^d - 2(2^{14}z - 1)^2$$

d even $\Rightarrow$ 4 real roots

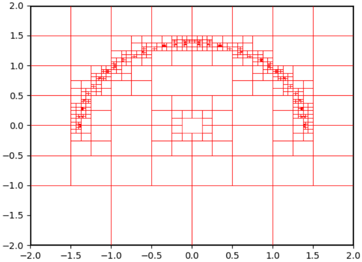
$d = 64$



Subdivision tree:



2044 T^0-tests



1072 T^0-tests (ratio $\simeq 0.52$)

Pols with real coefficients (II)

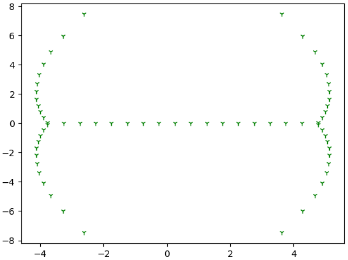
Example:

$\mathtt{Bern}_d(z) = \sum_{k=0}^{d} \binom{d}{k} b_{d-k} z^k$

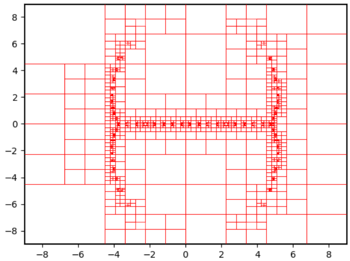
b_i's: Bernoulli numbers

d even $\Rightarrow d/4$ real roots

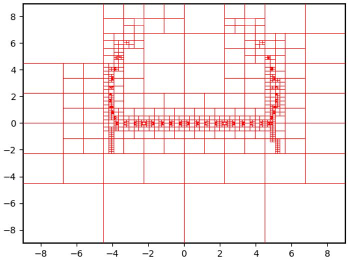
$d = 64$



Subdivision tree:



2492 T^0-tests



1476 T^0-tests (ratio $\simeq 0.6$)

Results (I)

Ccluster: version of [IPY18]
t_1: time; s_1: number of T^0-tests

CclusterR: Ccluster for polynomials in $\mathbb{R}[z]$
t_2: time; s_2: number of T^0-tests

	Ccluster			CclusterR	
	(#Clus, #Sols)	s_1	t_1	s_2	t_1/t_2
Bern_{128}	(128, 128)	4732	6.30	2712	1.72
Bern_{191}	(191, 191)	7220	20.2	4152	1.74
Bern_{256}	(256, 256)	9980	41.8	5698	1.67
Bern_{383}	(383, 383)	14504	120	8198	1.82
Mign_{128}	(127, 128)	4508	5.00	2292	1.92
Mign_{191}	(190, 191)	6260	15.5	3180	2.01
Mign_{256}	(255, 256)	8452	31.8	4304	2.04
Mign_{383}	(382, 383)	12564	79.7	6410	1.98

sequential times in s. on a Intel(R) Core(TM) i7-7600U CPU @ 2.80GHz machine with Linux

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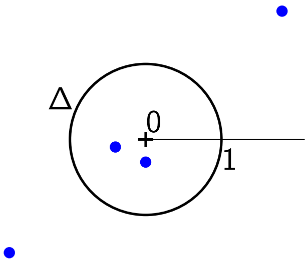
In *MACIS 19*, 2019.

Approximating Power Sums

Let $\Delta = \Delta(0,1)$, f has deg. d,
dist. roots $\alpha_1, \ldots, \alpha_{d_\Delta}$ in Δ with mults $m_1, \ldots, m_{d_\Delta}$

Power Sums: let $h \in \mathbb{Z}$

$$s_h = m_1 \times \alpha_1^h + \ldots + m_{d_\Delta} \times \alpha_{d_\Delta}^h$$



Approximating Power Sums

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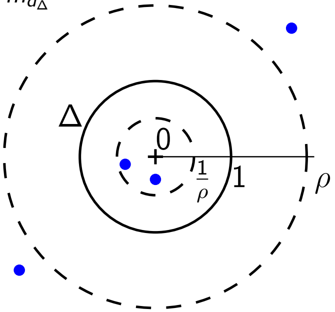


Theorem [S82, P18]:
if no root in $\{z \in \mathbb{C} | \frac{1}{\rho} < |z| < \rho\}$
use evaluations of f and f' at q points
to approximate s_h within error $\simeq d\rho^{-q}$

[Pan18] Victor Y Pan.
Old and new nearly optimal polynomial root-finders.
arXiv preprint arXiv:1805.12042, 2018.

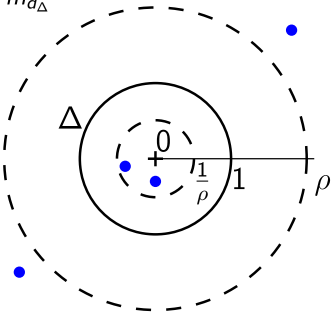
[Sch82] Arnold Schönhage.
The fundamental theorem of algebra in terms of computational complexity.
Manuscript. Univ. of Tübingen, Germany, 1982.

Approximating 0-th Power Sum

Let $\Delta = \Delta(0,1)$, f has deg. d,
dist. roots $\alpha_1, \ldots, \alpha_{d_\Delta}$ in Δ with mults $m_1, \ldots, m_{d_\Delta}$

Power Sums: let $h \in \mathbb{Z}$

$$s_0 = m_1 \times \alpha_1^0 + \ldots + m_{d_\Delta} \times \alpha_{d_\Delta}^0 = \#(\Delta, f)$$



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Approximating 0-th Power Sum

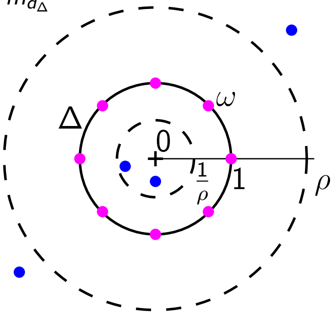
Let $\Delta = \Delta(0,1)$, f has deg. d,
dist. roots $\alpha_1, \ldots, \alpha_{d_\Delta}$ in Δ with mults $m_1, \ldots, m_{d_\Delta}$

0-th Power Sum:

$$s_0 = \#(\Delta, f)$$

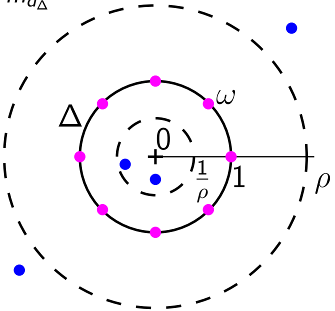
Approximation formula: let $q \in \mathbb{N}_*$, $\omega = e^{\frac{2\pi\imath}{q}}$

$$s_0^* = \frac{1}{q}\sum_{g=0}^{q-1} \omega^g \frac{f'(\omega^g)}{f(\omega^g)}$$



Approximating 0-th Power Sum

Let $\Delta = \Delta(0,1)$, f has deg. d,
dist. roots $\alpha_1, \dots, \alpha_{d_\Delta}$ in Δ with mults $m_1, \dots, m_{d_\Delta}$



0-th Power Sum:

$$s_0 = \#(\Delta, f)$$

Approximation formula: let $q \in \mathbb{N}_*$, $\omega = e^{\frac{2\pi\imath}{q}}$

$$s_0^* = \frac{1}{q}\sum_{g=0}^{q-1} \omega^g \frac{f'(\omega^g)}{f(\omega^g)}$$

Corollary of [S82, P18]: if no root in $\{z \in \mathbb{C} | \frac{1}{\rho} < |z| < \rho\}$, $\theta = 1/\rho$, then

(i) $|s_0^* - s_0| \leq \dfrac{d\theta^q}{1-\theta^q}$.

(ii) Fix $\delta > 0$. If $q = \lceil \log_\theta(\frac{\delta}{d+\delta}) \rceil$ then $|s_0^* - s_0| \leq \delta$.

Oracle numbers and polynomials

Let $\alpha \in \mathbb{C}$.

Oracle for α: function $\mathcal{O}_\alpha : \mathbb{Z} \rightarrow \square\mathbb{C}$
s.t. $\alpha \in \mathcal{O}_\alpha(L)$ and $w(\mathcal{O}_\alpha(L)) \leq 2^{-L}$

Let $f \in \mathbb{C}[z]$

Evaluation oracle for f: function $\mathcal{I}_f : \mathbb{Z} \times (\mathbb{Z} \rightarrow \square\mathbb{C}) \rightarrow \square\mathbb{C}$
s.t. $f(\alpha) \in \mathcal{I}_f(L, \mathcal{O}_\alpha)$ and $w(\mathcal{I}_f(L, \mathcal{O}_\alpha)) \leq 2^{-L}$

Notations: $\square\mathbb{C}$: set of complex interval
$\mathbb{Z} \rightarrow \square\mathbb{C}$: set of oracle numbers

The P^*-test

$P^*(\mathcal{I}_f, \mathcal{I}_{f'}, \Delta, \rho)$ *//Output in* $\{0, 1, \ldots, d\}$

Input: $\mathcal{I}_f, \mathcal{I}_{f'}$ evaluation oracles for f and f', Δ a disc ρ-isolated
Output: $\#(\Delta, f)$

1. $\delta \leftarrow 1/4$, $\theta \leftarrow 1/\rho$
2. $q \leftarrow \lceil \log_\theta(\frac{\delta}{d+\delta}) \rceil$
3.
4.
5.

The P^*-test

$P^*(\mathcal{I}_f, \mathcal{I}_{f'}, \Delta, \rho)$ *//Output in $\{0, 1, \ldots, d\}$*

Input: $\mathcal{I}_f, \mathcal{I}_{f'}$ evaluation oracles for f and f', Δ a disc ρ-isolated
Output: $\#(\Delta, f)$

1. $\delta \leftarrow 1/4$, $\theta \leftarrow 1/\rho$
2. $q \leftarrow \lceil \log_\theta(\frac{\delta}{d+\delta}) \rceil$
3. compute $\square s_0^*$ with $q, \mathcal{I}_f, \mathcal{I}_{f'}$ so that $w(\square s_0^*) < 1/2$
4.
5.

The P^*-test

$P^*(\mathcal{I}_f, \mathcal{I}_{f'}, \Delta, \rho)$ *//Output in $\{0, 1, \dots, d\}$*

Input: $\mathcal{I}_f, \mathcal{I}_{f'}$ evaluation oracles for f and f', Δ a disc ρ-isolated
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3. compute $\square s_0^*$ with $q, \mathcal{I}_f, \mathcal{I}_{f'}$ so that $w(\square s_0^*) < 1/2$
4. $\square s_0 \leftarrow \square s_0^* + [-1/4, 1/4] + \imath[-1/4, 1/4]$ // $w(\square s_0) < 1$
5.

The P^*-test

$P^*(\mathcal{I}_f, \mathcal{I}_{f'}, \Delta, \rho)$ *//Output in $\{0, 1, \dots, d\}$*

Input: $\mathcal{I}_f, \mathcal{I}_{f'}$ evaluation oracles for f and f', Δ a disc ρ-isolated
Output: $\#(\Delta, f)$

1. $\delta \leftarrow 1/4$, $\theta \leftarrow 1/\rho$
2. $q \leftarrow \lceil \log_\theta(\frac{\delta}{d+\delta}) \rceil$
3. compute $\Box s_0^*$ with $q, \mathcal{I}_f, \mathcal{I}_{f'}$ so that $w(\Box s_0^*) < 1/2$
4. $\Box s_0 \leftarrow \Box s_0^* + [-1/4, 1/4] + \imath[-1/4, 1/4]$ // $w(\Box s_0) < 1$
5. **return** the unique integer in $\Box s_0$

Example: f has degree 500, $\rho = 2$
evaluate f and f' at $q = 11$ points
then get $\#(\Delta, f)$ in $O(q)$ arithmetic operations

The P^*-test

$P^*(\mathcal{I}_f, \mathcal{I}_{f'}, \Delta, \rho)$ *//Output in* $\{0, 1, \dots, d\}$

Input: $\mathcal{I}_f, \mathcal{I}_{f'}$ evaluation oracles for f and f', Δ a disc ρ-isolated
Output: $\#(\Delta, f)$

1. $\delta \leftarrow 1/4$, $\theta \leftarrow 1/\rho$
2. $q \leftarrow \lceil \log_\theta(\frac{\delta}{d+\delta}) \rceil$
3. compute $\square s_0^*$ with $q, \mathcal{I}_f, \mathcal{I}_{f'}$ so that $w(\square s_0^*) < 1/2$
4. $\square s_0 \leftarrow \square s_0^* + [-1/4, 1/4] + \imath[-1/4, 1/4]$ // $w(\square s_0) < 1$
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Example: f has degree 500, $\rho = 2$
evaluate f and f' at $q = 11$ points
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Efficiency: directly related to evaluation

The P^*-test

	Discarding tests					
		T^*-tests		P^*-tests		
	nb	t_0	t_0/t (%)	t'_0	n_{-1}	n_{err}
$\mathtt{Bern}_{128}$	4732	5.50	86.9	1.38	269	10
$\mathtt{Bern}_{256}$	9980	36.3	87.8	7.61	561	20
$\mathtt{Mign}_{128}$	4508	4.73	90.9	0.25	276	12
$\mathtt{Mign}_{256}$	8452	27.8	91.2	0.60	544	20

P^*-tests: $P^*(\mathcal{I}_f, \mathcal{I}_{f'}, \Delta, 2)$
nb: nb of discarding tests performed
t: time in Ccluster
t_0: time in discarding T^*-tests
t'_0: time in P^*-tests

Example: f has degree 500, $\rho = 2$
evaluate f and f' at $q = 11$ points
then get $\#(\Delta, f)$ in $O(q)$ arithmetic operations

Efficiency: directly related to evaluation

The P^*-test

$P^*(\mathcal{I}_f, \mathcal{I}_{f'}, \Delta, \rho)$ *//Output in $\{0, 1, \dots, d\}$*

Input: $\mathcal{I}_f, \mathcal{I}_{f'}$ evaluation oracles for f and f', Δ a disc ρ-isolated
Output: $\#(\Delta, f)$

1. $\delta \leftarrow 1/4$, $\theta \leftarrow 1/\rho$
2. $q \leftarrow \lceil \log_\theta(\frac{\delta}{d+\delta}) \rceil$
3. compute $\Box s_0^*$ with $q, \mathcal{I}_f, \mathcal{I}_{f'}$ so that $w(\Box s_0^*) < 1/2$
4. $\Box s_0 \leftarrow \Box s_0^* + [-1/4, 1/4] + \imath[-1/4, 1/4]$ // $w(\Box s_0) < 1$
5. **return** the unique integer in $\Box s_0$

Example: f has degree 500, $\rho = 2$
evaluate f and f' at $q = 11$ points
then get $\#(\Delta, f)$ in $O(q)$ arithmetic operations

Efficiency: directly related to evaluation

But: requires ρ to be known and > 1.

The P^*-test

	Discarding tests					
		T^*-tests			P^*-tests	
	nb	t_0	t_0/t (%)	t'_0	n_{-1}	n_{err}
$\texttt{Bern}_{128}$	4732	5.50	86.9	1.38	269	10
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$\texttt{Mign}_{256}$	8452	27.8	91.2	0.60	544	20

P^*-tests: $P^*(\mathcal{I}_f, \mathcal{I}_{f'}, \Delta, 2)$

nb: nb of discarding tests performed

n_{-1}: nb of times □s_0 does not contains integer

n_{err}: nb of times result is not correct

Example: f has degree 500, $\rho = 2$
evaluate f and f' at $q = 11$ points
then get $\#(\Delta, f)$ in $O(q)$ arithmetic operations

Efficiency: directly related to evaluation

But: requires ρ to be known and > 1.

Using the P^*-test as a filter

The C^0-test:

$$C^0(\Delta) := \begin{cases} -1 & \text{if } P^*(\mathcal{I}_f, \mathcal{I}_{f'}, \Delta, 2) \neq 0, \\ -1 & \text{if } P^*(\mathcal{I}_f, \mathcal{I}_{f'}, \Delta, 2) = 0 \text{ and } T^*(\Delta, \mathcal{O}_f) \neq 0, \\ 0 & \text{if } P^*(\mathcal{I}_f, \mathcal{I}_{f'}, \Delta, 2) = 0 \text{ and } T^*(\Delta, \mathcal{O}_f) = 0. \end{cases}$$

Results (I)

Ccluster: version of [IPY18]
t_1: time; s_1: number of T^0-tests

CclusterR: Ccluster for polynomials in $\mathbb{R}[z]$
t_2: time; s_2: number of T^0-tests

CclusterP: CclusterR with P^*-test as a filter
t_3: time; s_3: number of T^0-tests

	Ccluster			CclusterR		CclusterP		
	(#Clus, #Sols)	s_1	t_1	s_2	t_1/t_2	s_3	t_3	t_2/t_3
Bern_{128}	(128, 128)	4732	6.30	2712	1.72	1983	3.30	1.10
Bern_{191}	(191, 191)	7220	20.2	4152	1.74	3073	10.7	1.08
Bern_{256}	(256, 256)	9980	41.8	5698	1.67	4067	21.9	1.14
Bern_{383}	(383, 383)	14504	120	8198	1.82	5813	53.5	1.23
Mign_{128}	(127, 128)	4508	5.00	2292	1.92	1668	1.81	1.43
Mign_{191}	(190, 191)	6260	15.5	3180	2.01	2431	4.34	1.77
Mign_{256}	(255, 256)	8452	31.8	4304	2.04	3223	10.7	1.44
Mign_{383}	(382, 383)	12564	79.7	6410	1.98	4883	26.8	1.49

sequential times in s. on a Intel(R) Core(TM) i7-7600U CPU @ 2.80GHz machine with Linux

Pols with real coefficients

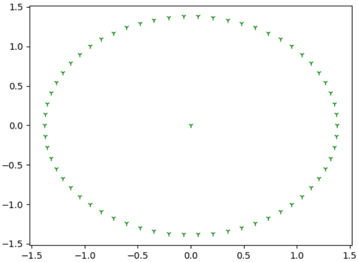
Example:

$\mathtt{Mign}_d(z) = z^d - 2(2^{14}z - 1)^2$

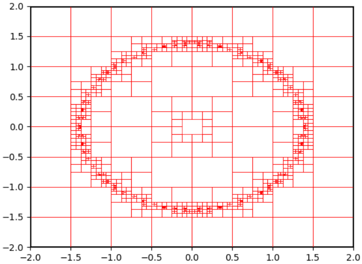
d even $\Rightarrow$ 4 real roots

only 4 non-zero coeffs

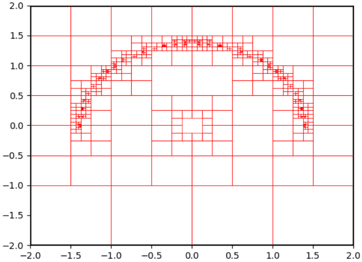
$d = 64$



Subdivision tree:



2044 T^0-tests

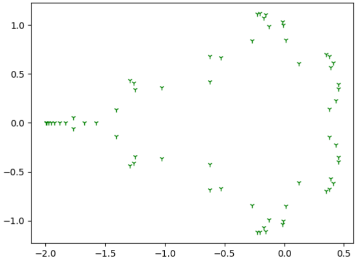


1072 T^0-tests (ratio $\simeq 0.52$)

Procedural polynomials

Procedure: $\mathtt{Mand}_k(z)$
Input: $k \in \mathbb{N}^*, z \in \mathbb{C}$
Output: $r \in \mathbb{C}$

1. **if** $k = 1$ **then**
2. **return** z
3. **else**
4. **return** $z\mathtt{Mand}_{k-1}(z)^2 + 1$



$k = 6$ (deg = 63)

Procedural polynomials

Procedure: $\texttt{Mand}_k(z)$
Input: $k \in \mathbb{N}^*, z \in \mathbb{C}$
Output: $r \in \mathbb{C}$

1. **if** $k = 1$ **then**
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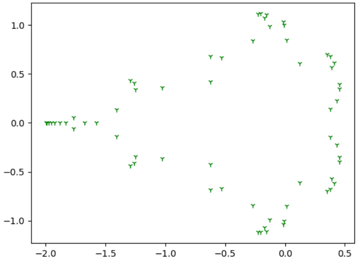
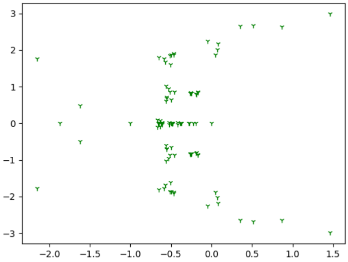


$k = 6$ (deg $= 63$)



$k = 8$ (deg $= 170$)

Procedure: $\texttt{Runn}_k(z)$
Input: $k \in \mathbb{N}, z \in \mathbb{C}$
Output: $r \in \mathbb{C}$

1. **if** $k = 0$ **then**
2. **return** 1
3. **else if** $k = 1$ **then**
4. **return** z
5. **else**
6. **return** $\texttt{Runn}_{k-1}(z)^2 + z\texttt{Runn}_{k-2}(z)^4$

Results (II)

Ccluster: version of [IPY18]
t_1: time

CclusterR: Ccluster for polynomials in $\mathbb{R}[z]$
t_2: time

CclusterP: CclusterR with P^*-test as a filter
t_3: time

	Ccluster		CclusterR	CclusterP		
	(#Clus, #Sols)	t_1	t_1/t_2	t_3	t_2/t_3	
$\texttt{Mand}_6$	(63, 63)	0.99	1.69	0.44	1.30	
$\texttt{Mand}_7$	(127, 127)	7.17	1.62	2.88	1.52	
$\texttt{Mand}_8$	(255, 255)	40.6	1.71	15.1	1.56	
$\texttt{Runn}_7$	(54, 85)	2.15	1.58	0.97	1.39	
$\texttt{Runn}_8$	(107, 170)	13.3	1.61	6.51	1.26	
$\texttt{Runn}_9$	(214, 341)	76.2	1.70	32.2	1.38	

sequential times in s. on a Intel(R) Core(TM) i7-7600U CPU @ 2.80GHz machine with Linux

Results (II)

Ccluster: version of [IPY18]
t_1: time

CclusterR: Ccluster for polynomials in $\mathbb{R}[z]$
t_2: time

CclusterP: CclusterR with P^*-test as a filter
t_3: time

	Ccluster		CclusterR	CclusterP		
	(#Clus, #Sols)	t_1	t_1/t_2	t_3	t_2/t_3	
$\mathtt{Mand}_6$	(63, 63)	0.99	1.69	0.44	1.30	
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$\mathtt{Runn}_9$	(214, 341)	76.2	1.70	32.2	1.38	

sequential times in s. on a Intel(R) Core(TM) i7-7600U CPU @ 2.80GHz machine with Linux

Results (II)

Ccluster: version of [IPY18]
t_1: time

ClusterR: Ccluster for polynomials in $\mathbb{R}[z]$
t_2: time

ClusterP: ClusterR with P^*-test as a filter
t_3: time

	Ccluster		ClusterR	ClusterP		MPSolve
	(#Clus, #Sols)	t_1	t_1/t_2	t_3	t_2/t_3	t_4
$\mathtt{Mand}_6$	(63, 63)	0.99	1.69	0.44	1.30	0.01
$\mathtt{Mand}_7$	(127, 127)	7.17	1.62	2.88	1.52	0.06
$\mathtt{Mand}_8$	(255, 255)	40.6	1.71	15.1	1.56	0.39
$\mathtt{Runn}_7$	(54, 85)	2.15	1.58	0.97	1.39	0.01
$\mathtt{Runn}_8$	(107, 170)	13.3	1.61	6.51	1.26	0.04
$\mathtt{Runn}_9$	(214, 341)	76.2	1.70	32.2	1.38	0.32

sequential times in s. on a Intel(R) Core(TM) i7-7600U CPU @ 2.80GHz machine with Linux

Thank you for your attention!