Numeric certified algorithm for computing the topology of projections of real space curves

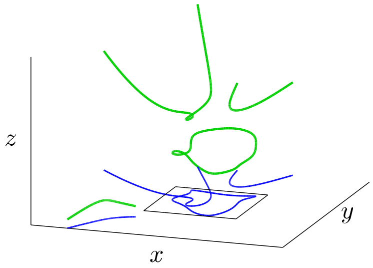
Rémi Imbach, Guillaume Moroz and Marc Pouget

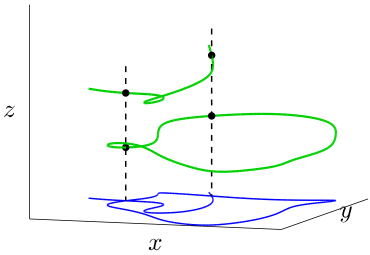
Projection and Apparent Contour

Curve defined as the intersection of two algebraic surfaces:

$$\mathcal{C} : \begin{cases} p(x,y,z) = 0 \\ q(x,y,z) = 0 \end{cases}, (x,y,z) \in \mathbb{R}^3$$

Projection in the plane: $\mathcal{B} = \pi_{(x,y)}(\mathcal{C})$



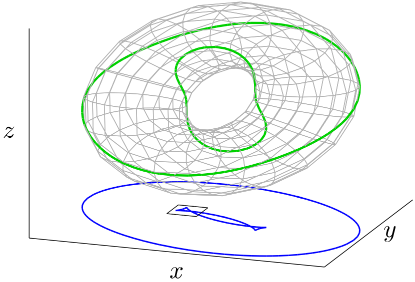


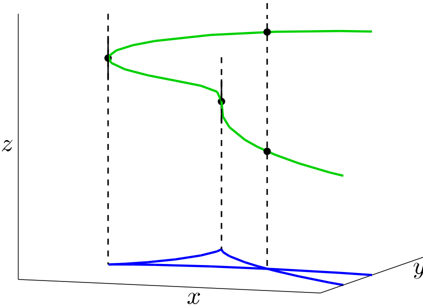
Projection and Apparent Contour

Curve defined as the intersection of two algebraic surfaces:

$$\mathcal{C} : \begin{cases} p(x,y,z) = 0 \\ p_z(x,y,z) = 0 \end{cases}, (x,y,z) \in \mathbb{R}^3, \qquad p_z = \frac{\partial p}{\partial z}$$

Apparent contour: $\mathcal{B} = \pi_{(x,y)}(\mathcal{C})$

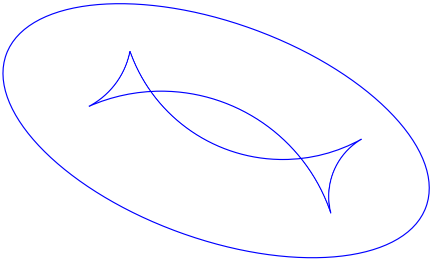


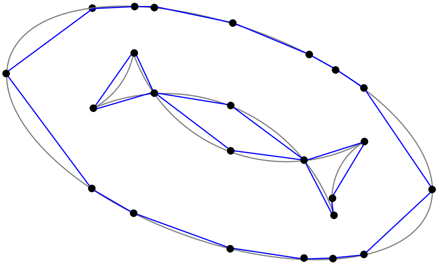


Computing topology of a real plane curve $\mathcal{B}$

$\mathcal{B} = \{(x, y) \in \mathbb{R}^2 | f(x, y) = 0\}$
Singularities: $\{(x, y) \in \mathbb{R}^2 | f(x, y) = f_x(x, y) = f_y(x, y) = 0\}$

- Path tracking methods fail near singularities





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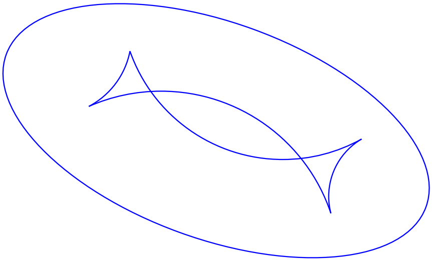
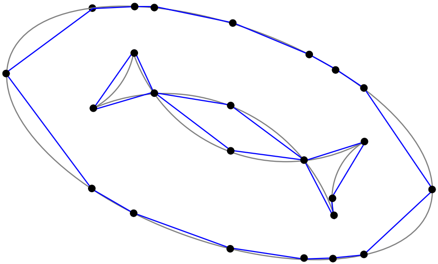
- Path tracking methods fail near singularities
- Symbolic methods
  - CAD requires : computing with algebraic numbers





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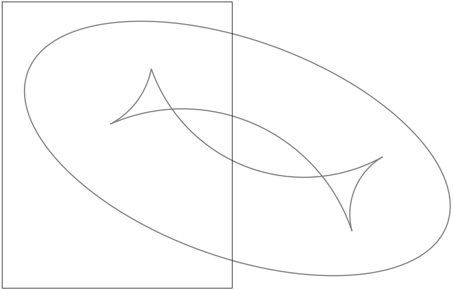

A general framework

0. Restrict to a compact $\mathbf{B}_0$
1. Isolate in boxes:
   - boundary points
   - x-critical points
   - singularities
2. Compute topology around singularities
3. Connect boxes

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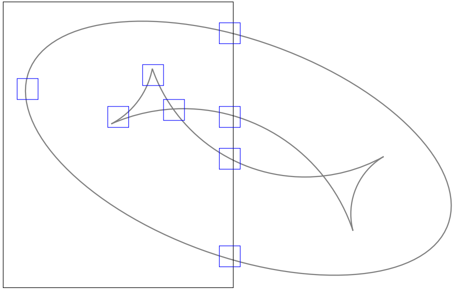

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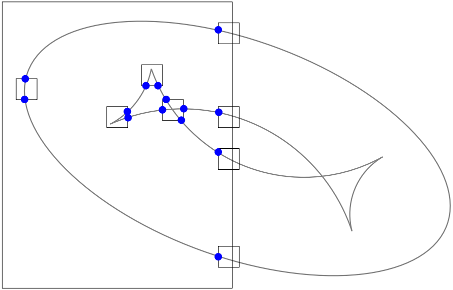

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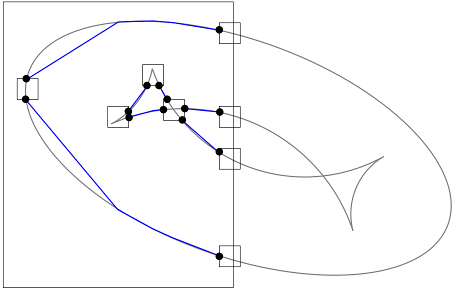

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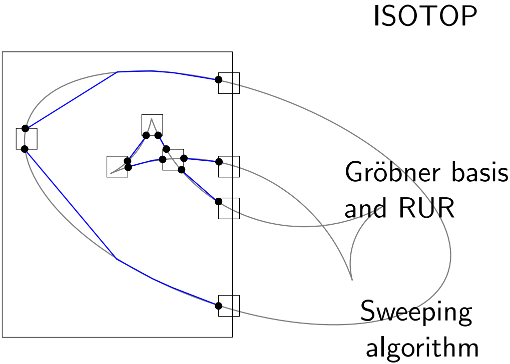

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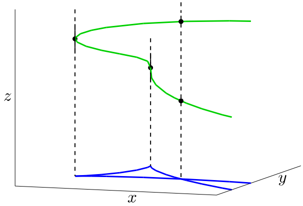
When $\mathcal{B}$ is a projection or an apparent contour



Geometric characterization of nodes and cusps:

- 4D square system
- 0-dim solver

❶ Isolate in boxes:

- boundary points
- x-critical points
- singularities

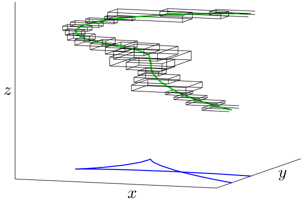
Certified numerical tools:

- 0-dim solver: subdivision

When $\mathcal{B}$ is a projection or an apparent contour

Enclosing $\mathcal{C}$ in a sequence of boxes:

- 1-dim solver
- 1 point on each C.C.: 0-dim solver



Geometric characterization of nodes and cusps:

- 4D square system
- 0-dim solver
- Restriction of the solving domain

❶ Isolate in boxes:

- boundary points
- x-critical points
- singularities

Certified numerical tools:

- 0-dim solver: subdivision
- 1-dim solver: path tracker

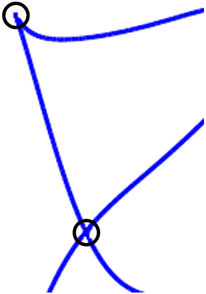
Isolating singularities

$\mathcal{B} = \{(x, y) \in \mathbb{R}^2 | r(x, y) = 0\}$,

Singularities of $\mathcal{B}$ are the solutions of:

$$\begin{cases} r(x,y) = 0 \\ \frac{\partial r}{\partial x}(x,y) = 0 \\ \frac{\partial r}{\partial y}(x,y) = 0 \end{cases}$$

. . . that is over-determined.



Isolating singularities

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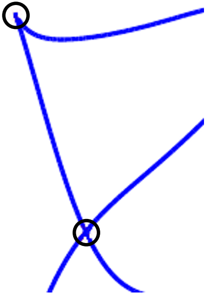
Singularities of $\mathcal{B}$ are the solutions of:

$$\begin{cases} r(x, y) = 0 \\ \frac{\partial r}{\partial x}(x, y) = 0 \end{cases} \text{s.t. } \frac{\partial r}{\partial y}(x, y) = 0$$

. . . that has solutions of multiplicity 2.



Isolating singularities of an apparent contour

$\mathcal{B} = \{(x,y) \in \mathbb{R}^2 | r(x,y) = 0\}$, where $r(x,y) = Res(p, p_z, z)(x,y)$

Singularities of $\mathcal{B}$ are the solutions of:

$$\begin{cases} r(x,y) = 0 \\ \frac{\partial r}{\partial x}(x,y) = 0 \end{cases} \quad \text{s.t. } \frac{\partial r}{\partial y}(x,y) = 0$$

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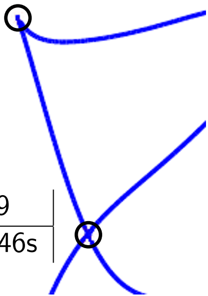
p,	degree 6,	bit-size 8,	84 monomials
r,	degree 30,	bit-size 111,	496 monomials
$\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}$,	degree 29,	bit-size 115,	465 monomials



degree of p	5	6	7	8	9
time with RSCube*	3.1s	32s	254s	1898s	9346s

* F.Rouillier

Isolating singularities of an apparent contour

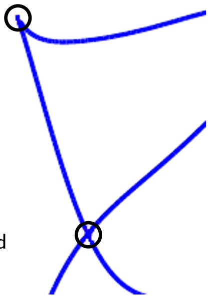
$\mathcal{B} = \{(x, y) \in \mathbb{R}^2 | r(x, y) = 0\}$, where $r(x, y) = Res(p, p_z, z)(x, y)$

Singularities of $\mathcal{B}$ are the regular solutions of:

$$(\mathcal{S}_2) \begin{cases} s_{10}(x, y) = 0 \\ s_{11}(x, y) = 0 \end{cases} \text{ s.t. } s_{22}(x, y) \neq 0$$

... where s_{10}, s_{11}, s_{22} are coefficients in the subresultant chain.

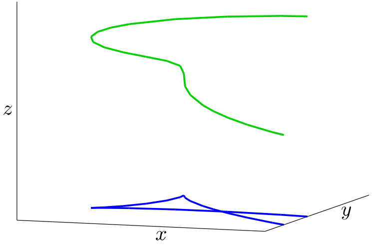
p,	degree 6,	bit-size 8,	84 monomials
r,	degree 30,	bit-size 111,	496 monomials
$\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}$,	degree 29,	bit-size 115,	465 monomials
s_{11}, s_{10},	degree 20,	bit-size 89,	231 monomials
s_{22},	degree 12,	bit-size 65,	91 monomials



[IMP15] Rémi Imbach, Guillaume Moroz, and Marc Pouget.
Numeric certified algorithm for the topology of resultant and discriminant curves.
Research Report RR-8653, Inria, April 2015.

Isolating singularities . . . : the ball system

$$\mathcal{B} = \{(x, y) \in \mathbb{R}^2 | \exists z \in \mathbb{R} \text{ s.t. } (x, y, z) \in \mathcal{C}\}$$



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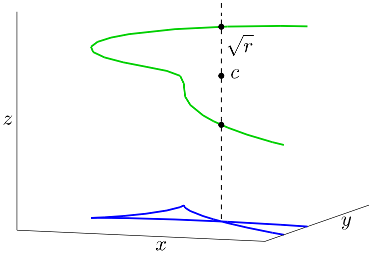

c: center of z_1, z_2
$r = \|cz_1\|_2^2$

Let (x, y) be:

- a node: $(x, y, z_1), (x, y, z_2) \in \mathcal{C}$, with $z_1 \neq z_2$

  $z_1 = c - \sqrt{r}$, $z_2 = c + \sqrt{r}$

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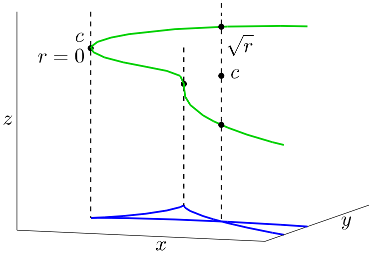

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  $z_1 = c - \sqrt{r}$, $z_2 = c + \sqrt{r}$
- a cusp: $(x, y, z_1), (x, y, z_2) \in \mathcal{C}$, with $z_1 = z_2$
  $z_1 = c - \sqrt{r}$, $z_2 = c + \sqrt{r}$

Isolating singularities . . . : the ball system

$$\mathcal{B} = \{(x, y) \in \mathbb{R}^2 | \exists z \in \mathbb{R} \text{ s.t. } (x, y, z) \in \mathcal{C}\}$$

Singularities of $\mathcal{B}$ are the regular solutions of:

$$(\mathcal{S}_4) \left\{ \begin{array}{rcll} \frac{1}{2}(p(x,y,c+\sqrt{r}) & + & p(x,y,c-\sqrt{r})) & = 0 \\ \frac{1}{2\sqrt{r}}(p(x,y,c+\sqrt{r}) & - & p(x,y,c-\sqrt{r})) & = 0 \\ \frac{1}{2}(p_z(x,y,c+\sqrt{r}) & + & p_z(x,y,c-\sqrt{r})) & = 0 \\ \frac{1}{2\sqrt{r}}(p_z(x,y,c+\sqrt{r}) & - & p_z(x,y,c-\sqrt{r})) & = 0 \end{array} \right.$$

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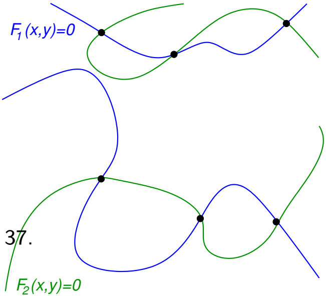
- equations of $(\mathcal{S}_4)$ are polynomials
- 4 equations in 4 unknowns

Certified numerical tools: 0-dim solver

$F : \mathbb{R}^n \rightarrow \mathbb{R}^n$, F polynomial,

- find zeros of F: find $\{X \in \mathbb{R}^n | F(X) = 0\}$

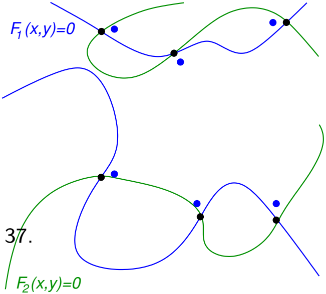


[Neu90] Arnold Neumaier.
Interval methods for systems of equations, volume 37.
Cambridge university press, 1990.

Certified numerical tools: 0-dim solver

$F : \mathbb{R}^n \rightarrow \mathbb{R}^n$, F polynomial,

- find zeros of F: find $\{X \in \mathbb{R}^n | F(X) = 0\} \rightsquigarrow \{X \in \mathbb{R}^n | \|F(X)\| \leq \epsilon\}$



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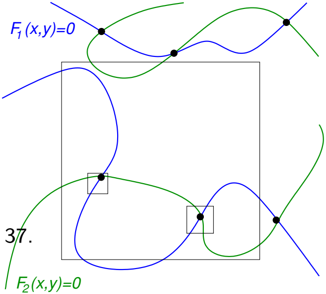
Certified numerical tools: 0-dim solver

$F : \mathbb{R}^n \rightarrow \mathbb{R}^n$, F polynomial, $\mathbf{X}_0$ a compact of $\mathbb{R}^n$

- find zeros of F: find $\{X \in \mathbb{R}^n | F(X) = 0\}$
- Isolate zeros of F in boxes $\{\mathbf{X}_1, \ldots, \mathbf{X}_n\}$ such that
  - each $\mathbf{X}_k$ contains a unique zero of F
  - each zero of F in $\mathbf{X}_0$ is in a unique box $\mathbf{X}_k$



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Interval Arithmetic: $\mathbf{x} \subset \mathbb{R}$,$\mathbf{X} \subset \mathbb{R}^n$

- multi-dimensional extension of interval : box $\mathbf{X} \subset \mathbb{R}^n$

$$\mathbf{X} = \mathbf{x_1} \times \ldots \times \mathbf{x_n} = [l(x_1), r(x_1)] \times \ldots \times [l(x_n), r(x_n)]$$

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Interval Arithmetic: $\mathbf{x} \subset \mathbb{R}$,$\mathbf{X} \subset \mathbb{R}^n$

- multi-dimensional extension of interval : box $\mathbf{X} \subset \mathbb{R}^n$
- interval arithmetic operators

$$\mathbf{x} = [l(x), r(x)], \mathbf{y} = [l(y), r(y)], \mathbf{x}+\mathbf{y} = [l(x)+l(y), r(x)+r(y)]$$

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- multi-dimensional extension of interval : box $\mathbf{X} \subset \mathbb{R}^n$
- interval arithmetic operators
- interval evaluation of $F : \mathbb{R}^n \to \mathbb{R}^n$: $F(\mathbf{X}) \supseteq \{F(X) | X \in \mathbf{X}\}$

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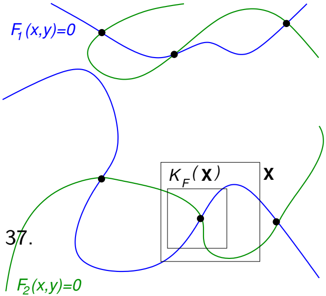
Krawczik criterion: $K_F : \mathbf{X} \subset \mathbb{R}^n \mapsto K_F(\mathbf{X}) \subset \mathbb{R}^n$

$K_F(\mathbf{X}) \subset Int(\mathbf{X}) \Rightarrow K_F(\mathbf{X})$ contains a unique zero of F

consequence of the Brouwer fixed point theorem.



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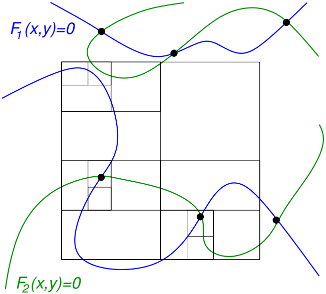
Krawczik criterion: $K_F : \mathbf{X} \subset \mathbb{R}^n \mapsto K_F(\mathbf{X}) \subset \mathbb{R}^n$

Subdivision method:

Input: $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\mathbf{X}_0$ box of $\mathbb{R}^n$
Output: A list R of boxes containing solutions in $\mathbf{X}_0$ of $F = 0$
$L := \{\mathbf{X}_0\}$
Repeat:
  $\mathbf{X} := L.pop$
  If $0 \in F(\mathbf{X})$ **then**
    If $K_F(\mathbf{X}) \subset Int(\mathbf{X})$ **then**
      insert $\mathbf{X}$ in R
    Else If $K_F(\mathbf{X}) \cap \mathbf{X} \neq \emptyset$ **then**
      bisect $\mathbf{X}$ and insert its sub-boxes in L
    End if
  End if
Until $L = \emptyset$
Return R



Certified numerical tools: 0-dim solver

$F : \mathbb{R}^n \rightarrow \mathbb{R}^n$, F polynomial, $\mathbf{X}_0$ a compact of $\mathbb{R}^n$

Interval Arithmetic: $\mathbf{x} \subset \mathbb{R}$,$\mathbf{X} \subset \mathbb{R}^n$,$F(\mathbf{X}) \supseteq \{F(X) | X \in \mathbf{X}\}$

Krawczik criterion: $K_F : \mathbf{X} \subset \mathbb{R}^n \mapsto K_F(\mathbf{X}) \subset \mathbb{R}^n$

Subdivision method:

- terminates with a correct result when
  - $F = 0$ has only regular solutions,
  - working at arbitrary precision.
- can be extended to unbounded initial box $\mathbf{X}_0$
- its cost grows exponentially with n

[Neu90] Arnold Neumaier.
Interval methods for systems of equations, volume 37.
Cambridge university press, 1990.

Certified numerical isolation of singularities

Datas: Random dense polynomials of degree d, bit-size 8

Subdivision solver: home made in `C++`, with `boost` interval library

- evaluation of polynomials with horner scheme $\rightarrow$ quick
- evaluation of polynomials at order 2 $\rightarrow$ sharp

Numerical results: Subdivision solving within $[-1, 1] \times [-1, 1]$

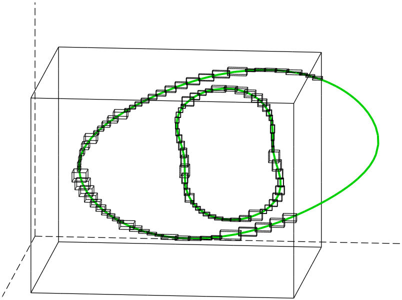
	Sub-resultant system $\mathcal{S}_2$	Ball system $\mathcal{S}_4$
d	t	t
5	0.05	24.8
6	0.50	8.40
7	4.44	43.8
8	37.9	70.2
9	23.1	45.6

means on 5 examples of sequential times in seconds on a Intel(R) Xeon(R) CPU L5640 @ 2.27GHz machine.

Restriction of the solving domain

Enclose $\mathcal{C}$: find a sequence $\{\mathbf{C}_k\}_{1 \leq k \leq l}$ such that

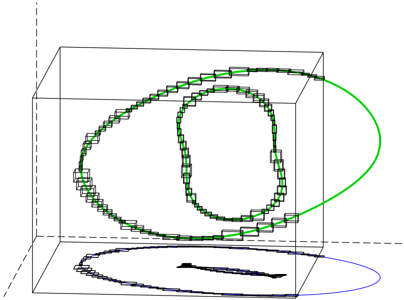
- $\mathcal{C} \subset \bigcup_k \mathbf{C}_k$,
- in each $\mathbf{C}_k$, $\mathcal{C} \cap \mathbf{C}_k$ is diffeomorphic to a close segment,
- each $\mathbf{C}_k$ has width less than η.



Restriction of the solving domain

Enclose $\mathcal{C}$: find a sequence $\{\mathbf{C}_k\}_{1\leq k\leq l}$ $\mathbf{C}_k = (\mathbf{x}_k, \mathbf{y}_k, \mathbf{z}_k)$

$\rightarrow$ Enclose $\mathcal{B}$: each $B \in \mathcal{B}$ is in a $\mathbf{B}_k = \pi_{(x,y)}(\mathbf{C}_k)$



Restriction of the solving domain

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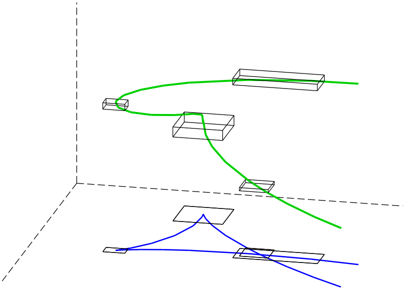
$\rightarrow$ Enclose singularities:

- each cusp is in a $\mathbf{B}_k$
- each node is in a $\mathbf{B}_{ij} = \mathbf{B}_i \cap \mathbf{B}_j$



Restriction of the solving domain

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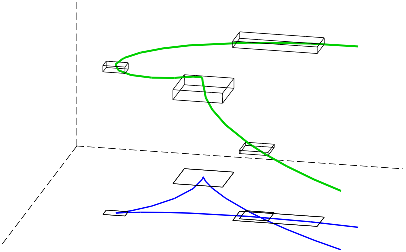
$\rightarrow$ Enclose $\mathcal{B}$: each $B \in \mathcal{B}$ is in a $\mathbf{B}_k = \pi_{(x,y)}(\mathbf{C}_k)$ $\mathbf{B}_k = (\mathbf{x}_k, \mathbf{y}_k)$

$\rightarrow$ Enclose singularities:

- each cusp is in a $\mathbf{B}_k$ $\mathbf{D}_k = (\mathbf{x}_k, \mathbf{y}_k, \mathbf{z}_k, [0, (\frac{w(\mathbf{z}_k)}{2})^2])$
- each node is in a $\mathbf{B}_{ij} = \mathbf{B}_i \cap \mathbf{B}_j$ $\mathbf{D}_{ij} = (\mathbf{x}_{ij}, \mathbf{y}_{ij}, \frac{(\mathbf{z}_i+\mathbf{z}_j)}{2}, [0, (\frac{(\mathbf{z}_i-\mathbf{z}_j)}{2})^2])$

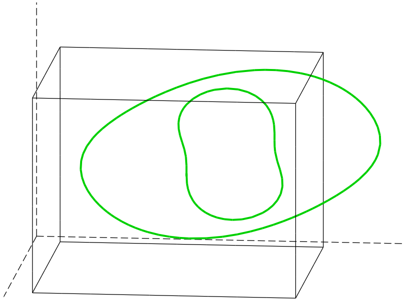
$\rightarrow$ Enclose solutions of the ball system:

Solutions of the ball system are in $\bigcup_k \mathbf{D}_k \cup \bigcup_{i,j} \mathbf{D}_{ij}$



Certified numerical tools: path tracker

$F : \mathbb{R}^n \to \mathbb{R}^{n-1}$, $\mathbf{X}_0$ a box of $\mathbb{R}^n$
$\mathcal{X} = \{X \in \mathbf{X}_0 | F(X) = 0\}$ is a smooth curve of $\mathbb{R}^n$
$\mathcal{X}^1, \ldots, \mathcal{X}^m$: connected components of $\mathcal{X}$



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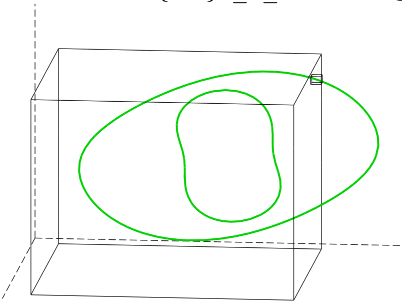
$\mathcal{X}^1, \ldots, \mathcal{X}^m$: connected components of $\mathcal{X}$

Certified path-tracker:

Input: $F : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$, $\mathbf{X}_0$ box of $\mathbb{R}^n$, $\eta \in \mathbb{R}^+_*$
An initial box $\mathbf{X} \in \mathcal{X}^i$

Output: a sequence of boxes $\{\mathbf{X}_k\}_{1 \leq k \leq l}$ enclosing $\mathcal{X}^i$.



Certified numerical tools: path tracker

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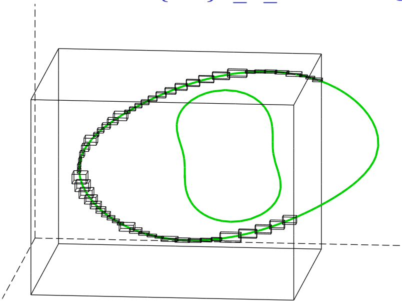
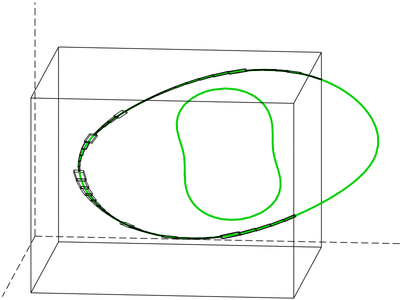


Certified numerical tools: path tracker

[MGGJ13] Benjamin Martin, Alexandre Goldsztejn, Laurent Granvilliers, and Christophe Jermann.
Certified parallelotope continuation for one-manifolds.
SIAM Journal on Numerical Analysis, 51(6):3373–3401, 2013.

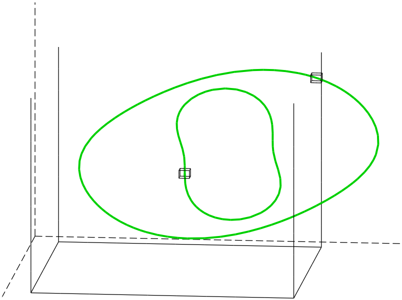


Enclosing $\mathcal{C}$

$F : \mathbb{R}^3 \rightarrow \mathbb{R}^2$, $\mathbf{B}_0$ a box of $\mathbb{R}^2$
$\mathcal{C} = \{C \in \mathbf{B}_0 \times \mathbb{R} | F(X) = 0\}$ is a smooth curve of $\mathbb{R}^3$
$\mathcal{C}^1, \ldots, \mathcal{C}^m$: connected components of $\mathcal{C}$

Assumption (A3): $\mathcal{C}$ is compact over $\mathbf{B}_0$
(A3) holds for generic polynomials p, q

Finding one point on each connected component

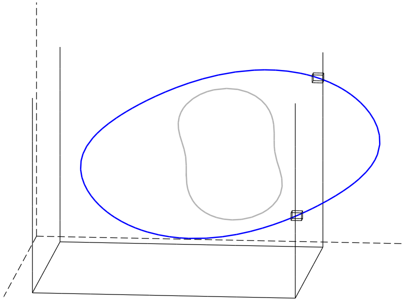


Finding one point on each connected component

Assumption (A3): $\mathcal{C}$ is compact over $\mathbf{B}_0$

Lemma: If (A3) holds, $\mathcal{C}^k$ is

- either diffeomorphic to $[0, 1]$
  $\Rightarrow$ has 2 intersections with $\partial \mathbf{B}_0 \times \mathbb{R}$
- or diffeomorphic to a circle
  $\Rightarrow$ has at least two x-critical points



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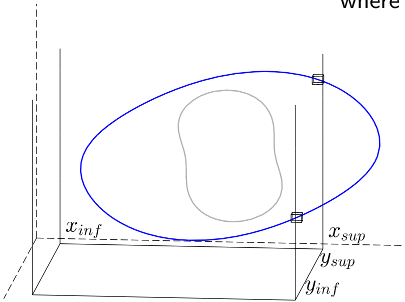
$\mathcal{C} \cap (\partial\mathbf{B}_0 \times \mathbb{R})$ are the solutions of the 4 systems:

$$\begin{cases} p(x = a, y, z) = 0 \\ q(x = a, y, z) = 0 \end{cases}$$

$$\begin{cases} p(x, y = b, z) = 0 \\ q(x, y = b, z) = 0 \end{cases}$$

where $a \in \{x_{inf}, x_{sup}\}$,
$b \in \{y_{inf}, y_{sup}\}$



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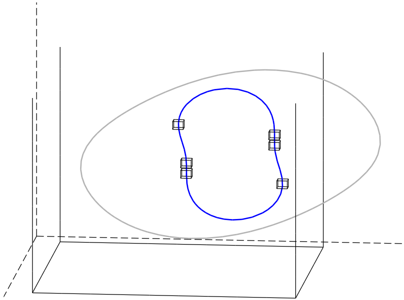

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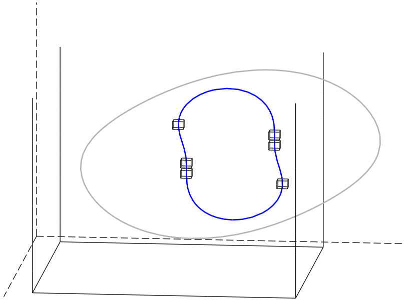
Assumption (A3): $\mathcal{C}$ is compact over $\mathbf{B}_0$

Lemma: If (A3) holds, $\mathcal{C}^k$ is

- either diffeomorphic to $[0,1]$
  $\Rightarrow$ has 2 intersections with $\partial \mathbf{B}_0 \times \mathbb{R}$
- or diffeomorphic to a circle
  $\Rightarrow$ has at least two x-critical points

x-critical points of $\mathcal{C}$ are the solutions of the system:

$$\begin{cases} p(x,y,z) & = 0 \\ q(x,y,z) & = 0 \\ \begin{vmatrix} p_y & p_z \\ q_y & q_z \end{vmatrix}(x,y,z) & = 0 \end{cases}$$



Certified numerical isolation of singularities

Path tracker: prototype in `python`/`cython`

Numerical results: solving within $[-1, 1] \times [-1, 1]$

d	Sub-resultant system $\mathcal{S}_2$ t	Ball system $\mathcal{S}_4$ t	$\mathcal{S}_4$ with curve tracking t
5	0.05	24.8	1.25
6	0.50	8.40	2.36
7	4.44	43.8	4.13
8	37.9	70.2	5.91
9	23.1	45.6	5.30

means on 5 examples of sequential times in seconds on a Intel(R) Xeon(R) CPU L5640 @ 2.27GHz machine.

Perspectives

- Using the enclosure of $\mathcal{C}$ to recover the topology of $\mathcal{B}$
- Projections of curves of $\mathbb{R}^n$, with $n > 3$

Questions?