Certified numerical tools for computing the topology of projected curves

Rémi Imbach, Guillaume Moroz and Marc Pouget

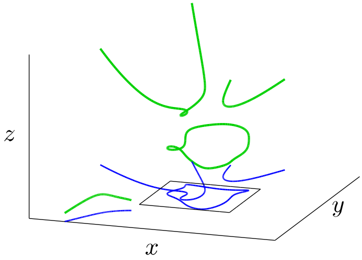
Projection and Apparent Contour

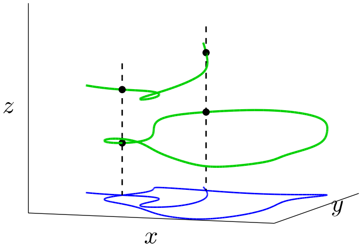
P, Q two (polynomial or analytic) maps $\mathbb{R}^3 \to \mathbb{R}$

Curve defined as the intersection of two surfaces:

$$\mathcal{C} : \begin{cases} P(x,y,z) &= 0 \\ Q(x,y,z) &= 0 \end{cases}, (x,y,z) \in \mathbb{R}^3$$

Projection in the plane: $\pi_{(x,y)}(\mathcal{C})$





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Curve defined as the intersection of two surfaces:

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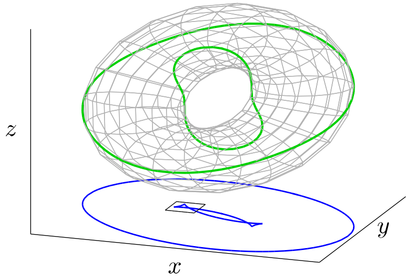
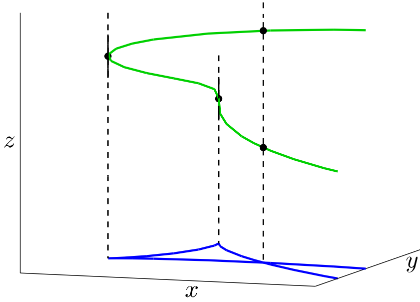
Apparent contour: $\pi_{(x,y)}(\mathcal{C})$



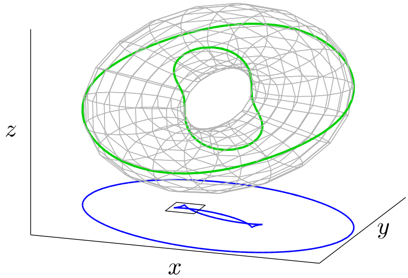


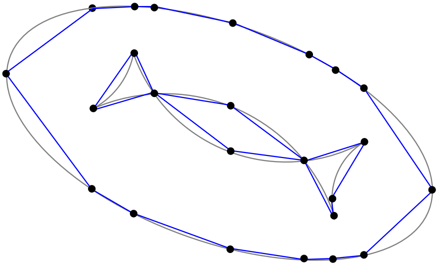
Computing the topology of a real plane curve $\mathcal{B}$

$$\mathcal{B} = \{(x, y) \in \mathbb{R}^2 | \exists z \in \mathbb{R} \text{ s.t. } (x, y, z) \in \mathcal{C}\}$$

Goal: with numerical approaches, compute

- exact topology
- approximated geometry





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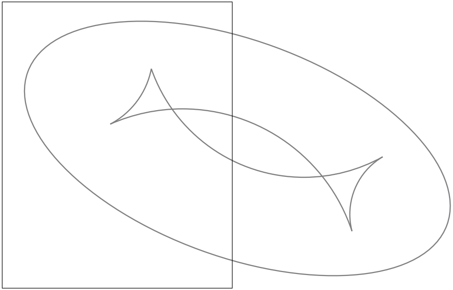

A general framework

0. Restrict to a compact $\mathbf{B}_0$
1. Isolate in boxes:
   - boundary points
   - x-critical points
   - singularities
2. Compute topology around singularities
3. Connect boxes

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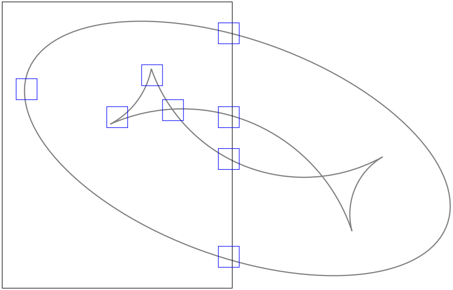

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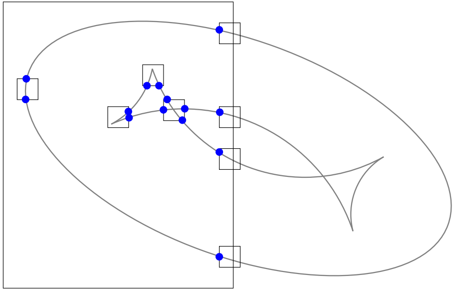

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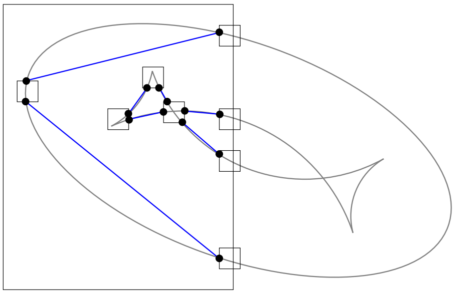

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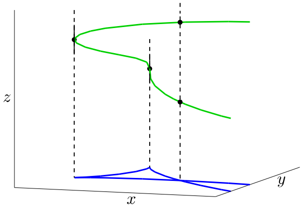
Computing the topology of a real plane curve $\mathcal{B}$

Characterization and isolation of nodes and cusps:

- Resultant approaches
- Geometric approach



❶ Isolate in boxes:

- boundary points
- x-critical points
- singularities

Certified numerical tools:

- 0-dim solver: branch and bound solver

Computing the topology of a real plane curve $\mathcal{B}$

Characterization and isolation of nodes and cusps:

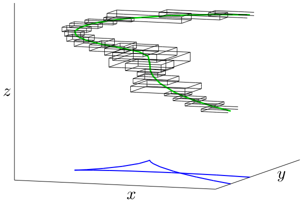
- Resultant approaches
- Geometric approach

Enclosing $\mathcal{C}$ in a sequence of boxes:

- Restrict the domain where singularities are sough
- Compute topology

Certified numerical tools:

- 0-dim solver: branch and bound solver
- 1-dim solver: certified path tracker



❶ Isolate in boxes:
- boundary points
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❷ Compute topology around singularities

❸ Connect boxes

Isolating singularities

$$\mathcal{B} = \{(x, y) \in \mathbb{R}^2 | r(x, y) = 0\},$$

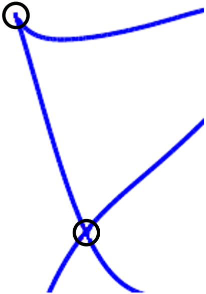
Singularities of $\mathcal{B}$ are the solutions of:

$$(\mathcal{S}) \begin{cases} r(x,y) = 0 \\ r_x(x,y) = 0 \\ r_y(x,y) = 0 \end{cases}$$

. . . that is over-determined

. . . that has solutions of multiplicity 2

symbolic approaches: Gröbner Basis, RUR



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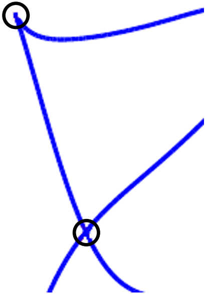

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Example

P, degree 6, bit-size 8, 84 monomials

$$P = 158x^6 - 186x^5y + 205x^5z - 160x^4y^2 + 105x^4yz + 116x^4z^2 - 69x^3y^3 - 161x^3y^2z - 8x^3yz^2 + 107x^3z^3 + 144x^2y^4 - 193x^2y^3z + 130x^2y^2z^2 + x^2yz^3 + 47x^2z^4 + 165xy^5 - 220xy^4z - 21xy^3z^2 + 50xy^2z^3 - 130xyz^4 - 77xz^5 + 66y^6 - 55y^5z + 219y^4z^2 - 30y^3z^3 - 162y^2z^4 - 182yz^5 - 145z^6 + 105x^5 + 241x^4y - 177x^4z - 127x^3y^2 - 97x^3yz + 223x^3z^2 - 46x^2y^3 - 213x^2y^2z + 39x^2yz^2 + 191x^2z^3 + 65xy^4 - 105xy^3z - 248xy^2z^2 + 158xyz^3 - 183xz^4 + 48y^5 - 240y^4z + 235y^3z^2 + 194y^2z^3 - 45yz^4 + 159z^5 - 81x^4 - 230x^3y - 247x^3z - 38x^2y^2 + 106x^2yz + 184x^2z^2 + 49xy^3 - 197xy^2z - 182xyz^2 - 223xz^3 - 205y^4 - 225y^3z - 14y^2z^2 - 17yz^3 + 73z^4 - 234x^3 - 82x^2y + 179x^2z + 46xy^2 - 222xyz - 95xz^2 + 139y^3 + 168y^2z + 8yz^2 + 156z^3 + 159x^2 - 147xy - 22xz - 104y^2 + 181yz + 26z^2 - 90x + 250y + 19z + 19$$

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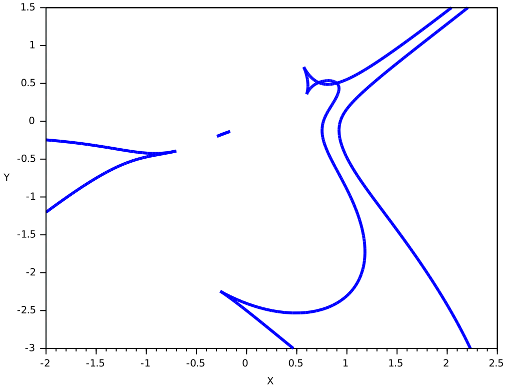
P, degree 6, bit-size 8, 84 monomials



Example

P,	degree 6,	bit-size 8,	84 monomials
r,	degree 30,	bit-size 111,	496 monomials

$$
\begin{aligned}
Res(P, P_z, z) = {} & 2537851751382193098537472618\,5 x^{30} - 19502895669848498217626626446\,0 x^{29} y + \\
& 66946066089386081392160455410\,0 x^{28} y^2 - 63132311630415225105620214800\,0 x^{27} y^3 - \\
& 102870456368043299024502235428\,0 x^{26} y^4 + 45977970156051179086240080820 x^{25} y^5 + \\
& 355446955340637129375198774227\,0 x^{24} y^6 + 371103101092844003966665661292\,0 x^{23} y^7 - \\
& 563444280018451438399891660026\,0 x^{22} y^8 - 1165859185506938114470659584106\,0 x^{21} y^9 - \\
& 438787493926607294806633245947\,0 x^{20} y^{10} + 164088434610382284202230231802\,30 x^{19} y^{11} + \\
& 2370016579425177706230400977291\,5 x^{18} y^{12} + 4316324180997748865901800201620 x^{17} y^{13} - \\
& 2492913730524765321908872849874\,0 x^{16} y^{14} - 3337290835102177803049211965481\,0 x^{15} y^{15} - \\
& 963344802815097587014751167457\,0 x^{14} y^{16} + 2050015543179023515840337400119\,0 x^{13} y^{17} + \\
& 3166808906075930935068471645835\,0 x^{12} y^{18} + 1654427855021865261625001839852\,0 x^{11} y^{19} - \\
& 501473052227565177171957565253\,5 x^{10} y^{20} - 1659011161494516371407397482332\,0 x^{9} y^{21} - \\
& 1354608334114918208346453586642\,5 x^{8} y^{22} - 475475994694179172456601211013\,0 x^{7} y^{23} + \\
& 1997218841447789708607357568825 x^{6} y^{24} + 389899802196825082224699960327\,0 x^{5} y^{25} +
\end{aligned}
$$

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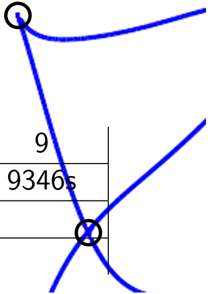


symbolic approaches: Gröbner Basis, RUR

degree of P	6	7	8	9
$(\mathcal{S})$ with `RSCube`*	32s	254s	1898s	9346s

* F. Rouillier

Sub-resultant based deflation system

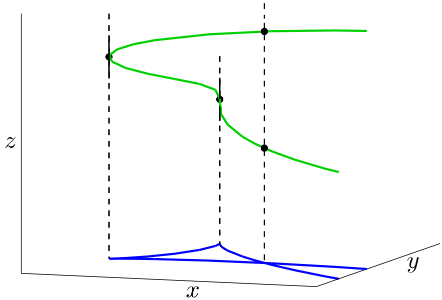
(α, β) node of $\mathcal{B}$

$\leftrightsquigarrow P(\alpha, \beta, z)$ and $P_z(\alpha, \beta, z)$ have two common roots z_0, z_1

(α, β) cusp of $\mathcal{B}$

$\leftrightsquigarrow P(\alpha, \beta, z)$ and $P_z(\alpha, \beta, z)$ have a double root z_0

$(\Rightarrow P_{zz}(\alpha, \beta, z_0) = 0)$



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$P(\alpha, \beta, z), P_z(\alpha, \beta, z)$ have a *gcd* of degree 2

Sub-resultant based deflation system

Sub-resultant chain of P, P_z, z:

$$\begin{array}{rlcccl} S^0 = & & & & & Res(P, P_z, z)(x,y) = r(x,y) \\ S^1 = & & & s_{11}(x,y)z & + & s_{10}(x,y) \\ S^2 = & s_{22}(x,y)z^2 & + & s_{21}(x,y)z & + & s_{20}(x,y) \\ \ldots = & & & \ldots & & \end{array}$$

where $s_{l,k} = det(A)$, $A \in \mathcal{M}_{(m+n-l)\times(m+n-l)}(\mathbb{Q}[x,y])$

Proposition $P(\alpha,\beta,z), P_z(\alpha,\beta,z)$ have a *gcd* of degree 2 iff $r(\alpha,\beta) = s_{11}(\alpha,\beta) = s_{10}(\alpha,\beta) = 0$ and $s_{22}(\alpha,\beta) \neq 0$.

Sub-resultant based deflation system

$$(\mathcal{S}_2)\begin{cases} s_{10}(x,y)=0 \\ s_{11}(x,y)=0 \quad \text{s.t.} \quad s_{22}(x,y)\neq 0 \\ r(x,y)=0 \end{cases}$$

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Remark Under genericity assumptions on P, one has:
if (x,y) is s.t. $s_{11}(x,y)=s_{10}(x,y)=0$ then
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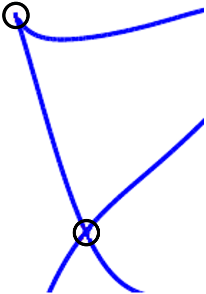

Example

P,	degree 6,	bit-size 8,	84 monomials
r,	degree 30,	bit-size 111,	496 monomials
s_{11}, s_{10},	degree 20,	bit-size 90,	231 monomials

$$s11 = -140117848627008812531220\,x^{20} - 610153133593349354171040\,x^{19}y + 395165189230217338440 70\,x^{18}y^2 + 334288372703346662015417 0\,x^{17}y^3 + 289127435514258940390189 0\,x^{16}y^4 + 112794729750527524649840\,x^{15}y^5 - 113406924905212987001252 20\,x^{14}y^6 - 110629111063889451654470 00\,x^{13}y^7 - 294644504237233492115385 0\,x^{12}y^8 + 128906414930624757578083 70\,x^{11}y^9 + 204828238814701231064683 70\,x^{10}y^{10} + 110248602292161309314200 10\,x^9y^{11} - 112696243429749597816286 0\,x^8y^{12} - 128844853246857476644326 80\,x^7y^{13} - 905972528707484832723458 0\,x^6y^{14} - 494132081742902565825385 0\,x^5y^{15} + 212239114641234869840676 0\,x^4y^{16} + 238411213685006877536954 0\,x^3y^{17} + 236334779693881164857826 0\,x^2y^{18} + 735933941537801203166720\,xy^{19} + 293011939904302120871210\,y^{20} + 693411445688541987909840\,x^{19} + 481966743447629919642227 0\,x^{18}y - 854531603999857310010090\,x^{17}y^2 - 458890306579609727152706 0\,x^{16}y^3 - 124545400776329858870419 90\,x^{15}y^4 - 190388099185807721139332 60\,x^{14}y^5 - 525559413440059828819296 0\,x^{13}y^6 + 117400526640477304407622 0\,x^{12}y^7 + 396580215854662355822437 20\,x^{11}y^8 + 491418220619801864690133 40\,x^{10}y^9 + 512514505112003918566666 90\,x^9y^{10} + 116693187859509164969230 50\,x^8y^{11} -$$

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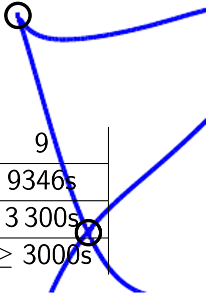

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$(\mathcal{S})$ with RSCube*	32s	254s	1898s	9346s
$(\mathcal{S}_2)$ with RSCube	15s	105s	620s	3 300s
$(\mathcal{S}_2)$ with Bertini	1005s	$\geq$ 3000s	$\geq$ 3000s	$\geq$ 3000s

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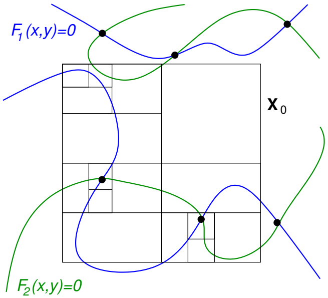
[IMP16] Rémi Imbach, Guillaume Moroz, and Marc Pouget.
A certified numerical algorithm for the topology of resultant and discriminant curves.
Journal of Symbolic Computation, 2016.

A branch and bound solver for systems of large polynomials



[Neu90] A. Neumaier.
Interval methods for systems of equations.
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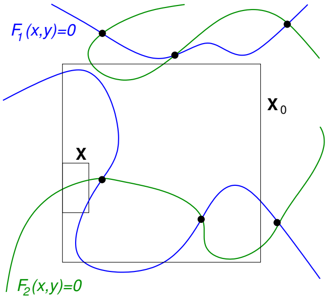

Interval extension $\Box F$ of F:

$0 \notin \Box F(\mathbf{X}) \Rightarrow$ no solution in $\mathbf{X}$

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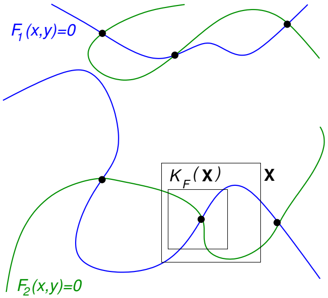

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Interval newton operators $K_F(\mathbf{X})$:

$K_F(\mathbf{X}) \subset int(\mathbf{X}) \Rightarrow \exists!$ solution in $\mathbf{X}$

- Interval Gauss-Seidel
- Krawczyk $K_F(\mathbf{X})$
- ...

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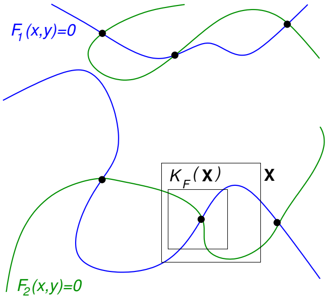
A branch and bound solver for systems of large polynomials



Interval extension $\square F$ of F:

$0 \notin \square F(\mathbf{X}) \Rightarrow$ no solution in $\mathbf{X}$

Interval newton operators $K_F(\mathbf{X})$:

$K_F(\mathbf{X}) \subset int(\mathbf{X}) \Rightarrow \exists!$ solution in $\mathbf{X}$

- Interval Gauss-Seidel
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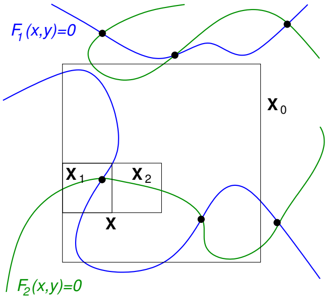

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Inclusion monotonicity:

$$\mathbf{X} = \mathbf{X}_1 \cup \mathbf{X}_2 \Rightarrow \square F(\mathbf{X}_1) \cup \square F(\mathbf{X}_2) \subseteq \square F(\mathbf{X})$$

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A branch and bound solver for systems of large polynomials

$s11 = -140117848627008812531220\,x^{20} - 610153133593349354171040\,x^{19}y +$

$39516518923021733844070\,x^{18}y^2 + 3342883727033466620154170\,x^{17}y^3 +$

$2891274355142589403901890\,x^{16}y^4 + 112794729750527524649840\,x^{15}y^5 -$

$11340692490521298700125220\,x^{14}y^6 - 11062911106388945165447000\,x^{13}y^7 -$

$2946445042372334921153850\,x^{12}y^8 + 12890641493062475757808370\,x^{11}y^9 +$

$20482823881470123106468370\,x^{10}y^{10} + 11024860229216130931420010\,x^9y^{11} -$

$1126962434297495978162860\,x^8y^{12} - 12884485324685747664432680\,x^7y^{13} -$

$9059725287074848327234580\,x^6y^{14} - 4941320817429025658253850\,x^5y^{15} +$

$2122391146412348698406760\,x^4y^{16} + 2384112136850068775369540\,x^3y^{17} +$

$2363347796938811648578260\,x^2y^{18} + 735933941537801203166720\,xy^{19} +$

$293011939904302120871210\,y^{20} + 693411445688541987909840\,x^{19} +$

$4819667434476299196422270\,x^{18}y - 854531603999857310010090\,x^{17}y^2 -$

$4588903065796097271527060\,x^{16}y^3 - 12454540077632985887041990\,x^{15}y^4 -$

$19038809918580772113933260\,x^{14}y^5 - 5255594134400598288192960\,x^{13}y^6 +$

$1174005266404773044076220\,x^{12}y^7 + 39658021585466235582243720\,x^{11}y^8 +$

$49141822061980186469013340\,x^{10}y^9 + 51251450511200391856666690\,x^9y^{10} +$

$11669318785950916496923050\,x^8y^{11} - 28301495536506792226988080\,x^7y^{12} -$

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Main Issues:

Evaluating F:

- quickly
- sharply

Adapting arithmetic precision

Evaluating large multivariate polynomials

Quickly: multivariate Horner scheme

Sharply: $\square F$ can be implemented by:

^0F: Horner form, linearly convergent toward F
^1F: centered eval. at order 1, quadratically convergent toward F
^2F: centered eval. at order 2, at least quadratically convergent toward F

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Pols	random			
d, σ	32, 32	64, 64	128, 128	
${}^{0}F$	1 188	1 503	1 730	
${}^{1}F$	1 054	1 293	1 432	
${}^{2}F$	747	966	952	

Nb of explored boxes and times in s., systems of 2 pols in 2 unknowns

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Pols	random						
d, σ	32, 32		64, 64		128, 128		
${}^0F$	1 188	0.17s	1 503	0.77s	1 730	3.8s	
${}^1F$	1 054	0.11s	1 293	0.67s	1 432	3.2s	
${}^2F$	747	0.09s	966	0.44s	952	1.9s	

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Pols	random						disc deg 8	
d, σ	32, 32		64, 64		128, 128		42, 128	
${}^0F$	1 188	0.17s	1 503	0.77s	1 730	3.8s	182 649	64.6s
${}^1F$	1 054	0.11s	1 293	0.67s	1 432	3.2s	134 337	44.2s
${}^2F$	747	0.09s	966	0.44s	952	1.9s	18 138	4.75s

Nb of explored boxes and times in s., systems of 2 pols in 2 unknowns

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Adapting arithmetic precision

Arithmetic precision is doubled if

Criteria of [Rev03]: $\{\mathbf{X}_1, \mathbf{X}_2\} = bisect(\mathbf{X})$, $w(\mathbf{X})$: width of $\mathbf{X}$

$(c1)$ the width of $\mathbf{X}$ is near the actual machine ϵ:

$$w(\mathbf{X}_1) \geq w(\mathbf{X}) \text{ or } w(\mathbf{X}_2) \geq w(\mathbf{X})$$

$(c2)$ $\square F(\mathbf{X})$ is no more inclusion monotonic

$$\square F(\mathbf{X}_1) \cup \square F(\mathbf{X}_2) \not\subseteq \square F(\mathbf{X})$$

[Rev03] N. Revol.
Interval newton iteration in multiple precision for the univariate case.
Numerical Algorithms, 34(2-4):417–426, 2003.

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Heuristic criterion for krawczyk operator: $w(\mathbf{X})$: width of $\mathbf{X}$

$$K_F(\mathbf{X}) = \mathbf{P} - J_F(\mathbf{P})^{-1}F(\mathbf{P}) + \Box J_F(\ldots), \text{ where } \mathbf{P} \text{ is a point in } \mathbf{X}$$

Certificate of existence and uniqueness only if $K_F(\mathbf{X}) \subset int(\mathbf{X})$

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$$\Rightarrow w(K_F(\mathbf{X})) = \underbrace{w(\mathbf{P} - J_F(\mathbf{P})^{-1}F(\mathbf{P}))}_{\leq \epsilon} + w(\square J_F(\ldots))$$

Certificate of existence and uniqueness only if $w(K_F(\mathbf{X})) < w(\mathbf{X})$

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$$K_F(\mathbf{X}) = \mathbf{P} - J_F(\mathbf{P})^{-1}F(\mathbf{P}) + \Box J_F(\ldots), \text{ where } \mathbf{P} \text{ is a point in } \mathbf{X}$$

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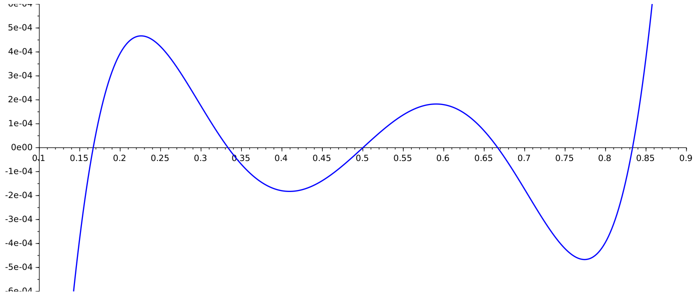

$F(X) = (X - \frac{1}{6}) \ldots (X - \frac{5}{6})$
mantissa: 15 bits
machine ϵ: $\simeq 10^{-4}$

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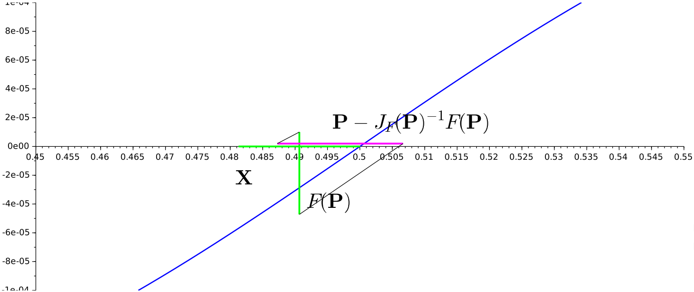

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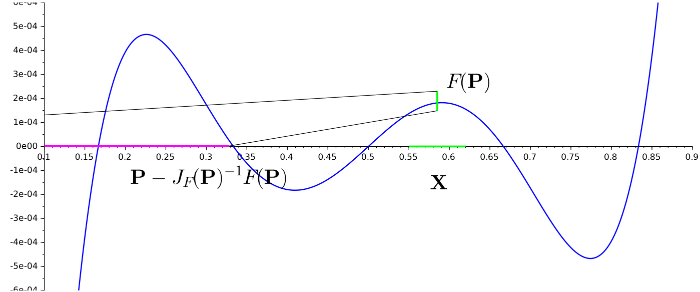

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$W_N(X) = (X - \frac{1}{N+1}) \dots (X - \frac{N}{N+1})$

	$W_{20}(X)$ on $[0,1]$			disc. deg 8 on $[-1,1] \times [-1,1]$		
	prec		t. in s.	prec		t. in s.
	53	106		53	106	
$(c2)$	519 983	129 297	$\simeq 8$	874 620	30 697	$\simeq 400$
$(c3)$	308 941	202 123	$\simeq 10$	723 146	3 013	$\simeq 207$

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Results:

Datas: Random dense polynomials of degree d, bit-size 8

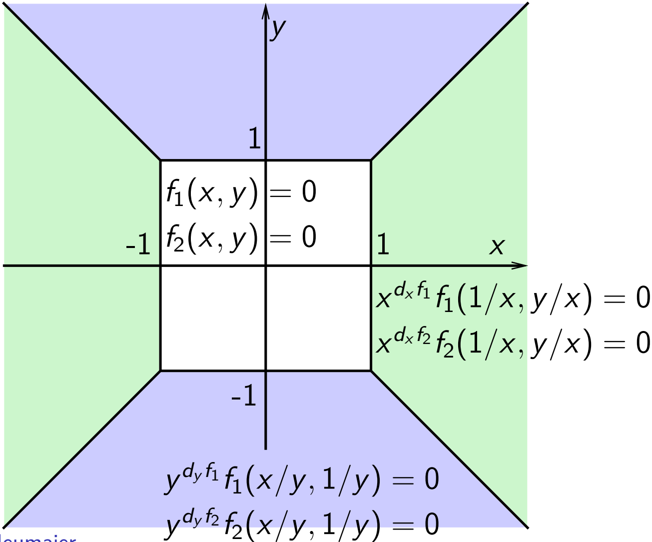
0-dim solver: multi-precision subdivision solver, `c++/cython/sage`
IA libraries: `BOOST` for double precision, `MPFI` otherwise

[Imb16] Rémi Imbach.
A Subdivision Solver for Systems of Large Dense Polynomials.
Technical Report 476, INRIA Nancy, March 2016.

Numerical results: Isolating singularities of an apparent contour

system domain d	$\mathcal{S}_2$, RSCube $\mathbb{R}^2$	$\mathcal{S}_2$, subd. $[-1,1]\times[-1,1]$	
6	15	0.5	
7	105	4.44	
8	620	37.9	
9	3300	23.2	

means on 5 examples of sequential times.



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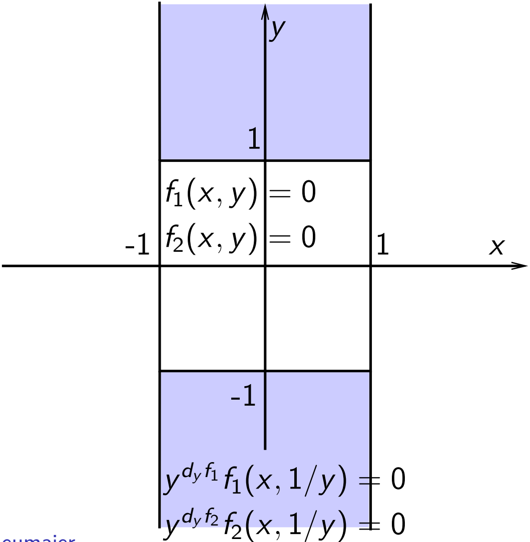

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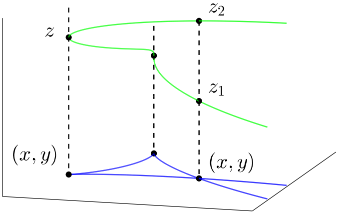
system	$\mathcal{S}_2$, RSCube	$\mathcal{S}_2$, subd.		
domain	$\mathbb{R}^2$	$[-1,1]\times[-1,1]$	$\mathbb{R}^2$	
d				
6	15	0.5	1.35	
7	105	4.44	21.9	
8	620	37.9	57.7	
9	3300	23.2	54.7	

means on 5 examples of sequential times.

Characterizing singularities:

$\mathcal{B} = \{(x,y) \in \mathbb{R}^2 | \exists z \in \mathbb{R} \text{ s.t. } (x,y,z) \in \mathcal{C}\}$



Lemma 1: (x, y) is a node of $\mathcal{B} \Leftrightarrow (x, y, z_1, z_2)$ satisfies:

$$P(x,y,z_1) = Q(x,y,z_1) = P(x,y,z_2) = Q(x,y,z_2) = 0$$

Lemma 2: (x, y) is a cusp of $\mathcal{B} \Leftrightarrow (x, y, z)$ satisfies:

$$P(x,y,z) = Q(x,y,z) = P_z(x,y,z) = Q_z(x,y,z) = 0$$

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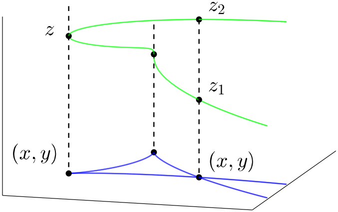

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Lemma 2: (x, y) is a cusp of $\mathcal{B} \Leftrightarrow (x, y, z)$ satisfies:

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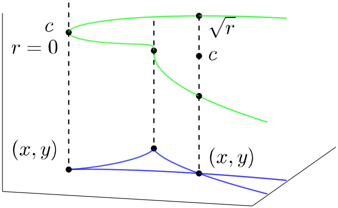
$$P(x, y, z) = P_z(x, y, z) = P_{zz}(x, y, z) = 0$$

Characterizing singularities: the Ball system

$$\mathcal{B} = \{(x, y) \in \mathbb{R}^2 | \exists z \in \mathbb{R} \text{ s.t. } (x, y, z) \in \mathcal{C}\}$$



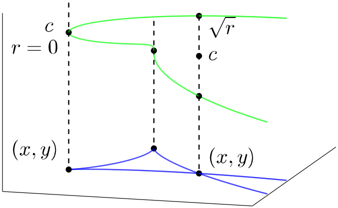
c: center of z_1, z_2
$r = \|cz_1\|_2^2$

Singularities of $\mathcal{B}$ are exactly the real solutions of:

$$(\mathcal{S}_4)\begin{cases} \frac{1}{2}(P(x,y,c+\sqrt{r}) + P(x,y,c-\sqrt{r})) & = 0 \\ \frac{1}{2\sqrt{r}}(P(x,y,c+\sqrt{r}) - P(x,y,c-\sqrt{r})) & = 0 \\ \frac{1}{2}(Q(x,y,c+\sqrt{r}) + Q(x,y,c-\sqrt{r})) & = 0 \\ \frac{1}{2\sqrt{r}}(Q(x,y,c+\sqrt{r}) - Q(x,y,c-\sqrt{r})) & = 0 \end{cases}$$

Characterizing singularities: the Ball system

$\mathcal{B} = \{(x,y) \in \mathbb{R}^2 | \exists z \in \mathbb{R} \text{ s.t. } (x,y,z) \in \mathcal{C}\}$



c: center of z_1, z_2
$r = \|cz_1\|_2^2$

Singularities of $\mathcal{B}$ are exactly the real solutions of:

when $r \to 0$

$$(\mathcal{S}_4) \begin{cases} P(x,y,c) &= 0 \\ P_z(x,y,c) &= 0 \\ Q(x,y,c) &= 0 \\ Q_z(x,y,c) &= 0 \end{cases}$$

Characterizing singularities: the Ball system

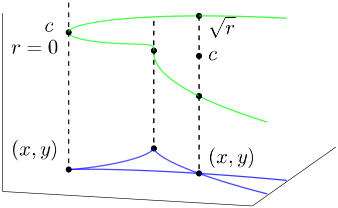
[IMP15] Rémi Imbach, Guillaume Moroz, and Marc Pouget.
Numeric and certified isolation of the singularities of the projection of a smooth space curve.
In *Proceedings of the 6th International Conferences on Mathematical Aspects of Computer and Information Sciences*, MACIS'15, 2015.

Lemma 4. Under some genericity assumptions, all the solutions of $\mathcal{S}_4$ in $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^+$ are regular.

Lemma 3. Singularities of $\mathcal{B}$ are exactly the real solutions of:

$$(\mathcal{S}_4) \begin{cases} \frac{1}{2}(P(x,y,c+\sqrt{r}) + P(x,y,c-\sqrt{r})) & = 0 \\ \frac{1}{2\sqrt{r}}(P(x,y,c+\sqrt{r}) - P(x,y,c-\sqrt{r})) & = 0 \\ \frac{1}{2}(Q(x,y,c+\sqrt{r}) + Q(x,y,c-\sqrt{r})) & = 0 \\ \frac{1}{2\sqrt{r}}(Q(x,y,c+\sqrt{r}) - Q(x,y,c-\sqrt{r})) & = 0 \end{cases}$$

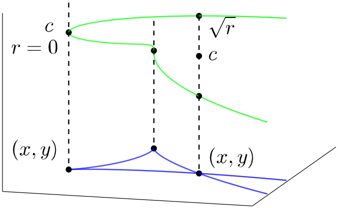
Isolating singularities: solving the Ball system



Finding the singularities of $\mathcal{B}$ in $\mathbf{B}_0 = (\mathbf{x}_0, \mathbf{y}_0)$:

$\leftrightsquigarrow$ solving the ball system on $\mathbf{B}_0 \times \mathbb{R} \times \mathbb{R}^+$

Isolating singularities: solving the Ball system



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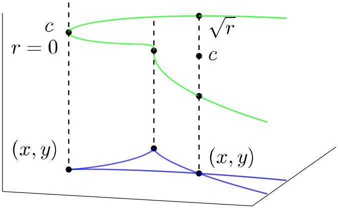
Isolating singularities: solving the Ball system



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Finding the singularities of $\mathcal{B}$ in $\mathbf{B} = \mathbb{R}^2$:

$\leftrightsquigarrow$ 5 systems solving on $[-1, 1]^2 \times [-1, 1] \times [0, 1]$

Results:

Datas: Random dense polynomials of degree d, bit-size 8

0-dim solver: multi-precision subdivision solver, `c++/cython/sage`
IA libraries: `BOOST` for double precision, `MPFI` otherwise

[Imb16] Rémi Imbach.
A Subdivision Solver for Systems of Large Dense Polynomials.
Technical Report 476, INRIA Nancy, March 2016.

Numerical results: Isolating singularities of an apparent contour

system domain d	$\mathcal{S}_2$, RSCube $\mathbb{R}^2$	$\mathcal{S}_2$, subd. $[-1,1]\times[-1,1]$	 $\mathbb{R}^2$	$\mathcal{S}_4$, subd. $[-1,1]\times[-1,1]$	 $\mathbb{R}^2$
6	15	0.5	1.35	8.4	11.3
7	105	4.44	21.9	43.8	54.2
8	620	37.9	57.7	70.2	99.2
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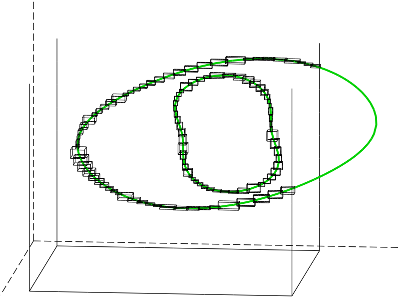
means on 5 examples of sequential times.

Computing the topology of $\mathcal{B}$: a geometric approach

Enclose $\mathcal{C}$: find a sequence $\{\mathbf{C}_k\}_{1 \leq k \leq l}$ such that

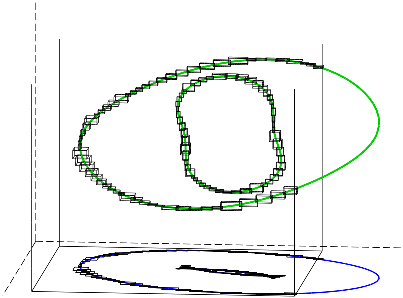
- $\mathcal{C} \subset \bigcup_k \mathbf{C}_k$,
- in each $\mathbf{C}_k$, $\mathcal{C} \cap \mathbf{C}_k$ is diffeomorphic to a close segment,
- each $\mathbf{C}_k$ has width less than η.



Computing the topology of $\mathcal{B}$: a geometric approach

Enclose $\mathcal{C}$: $\{\mathbf{C}_k\}_{1\leq k\leq l} = \{(\mathbf{x}_k, \mathbf{y}_k, \mathbf{z}_k)\}_{1\leq k\leq l}$

$\rightarrow$ Enclose $\mathcal{B}$: each $B \in \mathcal{B}$ is in a $\mathbf{B}_k = \pi_{(x,y)}(\mathbf{C}_k)$



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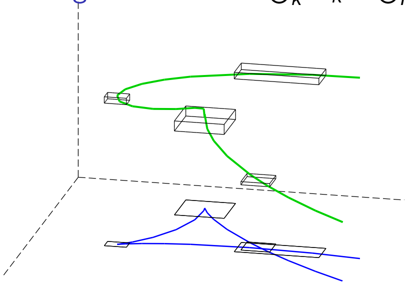
Enclose $\mathcal{B}$: $\{\mathbf{B}_k\}_{1\leq k\leq l} = \{(\mathbf{x}_k, \mathbf{y}_k)\}_{1\leq k\leq l}$

$\rightarrow$ Isolate singularities:

- each cusp is in a $\mathbf{B}_k$
- each node is in a $\mathbf{B}_{ij} = \mathbf{B}_i \cap \mathbf{B}_j$

$\rightarrow$ Singularities are in $\bigcup_k \mathbf{B}_k \cup \bigcup_{i,j} \mathbf{B}_{ij}$



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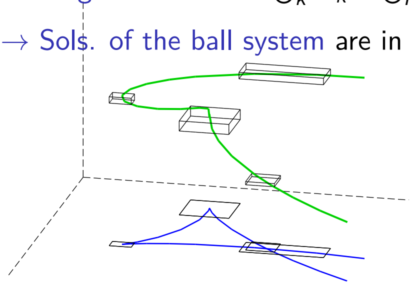
→ Isolate singularities:

- each cusp is in a $\mathbf{B}_k$ $\quad \mathbf{D}_k = (\mathbf{x}_k, \mathbf{y}_k, \mathbf{z}_k, [0, (\frac{w(\mathbf{z}_k)}{2})^2])$
- each node is in a $\mathbf{B}_{ij} = \mathbf{B}_i \cap \mathbf{B}_j$ $\quad \mathbf{D}_{ij} = (\mathbf{x}_{ij}, \mathbf{y}_{ij}, \frac{(\mathbf{z}_i+\mathbf{z}_j)}{2}, [0, (\frac{(\mathbf{z}_i-\mathbf{z}_j)}{2})^2])$

→ Singularities are in $\bigcup_k \mathbf{B}_k \cup \bigcup_{i,j} \mathbf{B}_{ij}$

→ Sols. of the ball system are in $\bigcup_k \mathbf{D}_k \cup \bigcup_{i,j} \mathbf{D}_{ij}$



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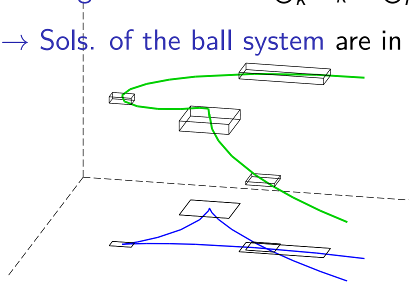

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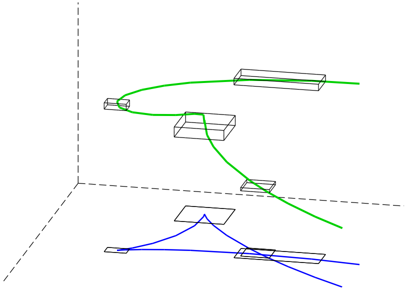
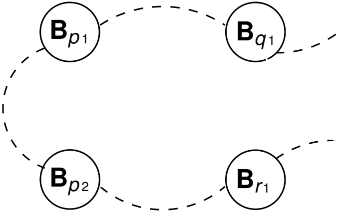
$\rightarrow$ Compute a graph:

- $\mathcal{G}_{\mathcal{B}} = (\{\mathbf{B}_k\}_{1\leq k\leq l}, \{(\mathbf{B}_k, \mathbf{B}_{k+1})\}_{1\leq k<l})$





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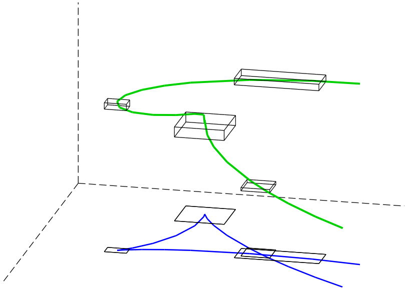
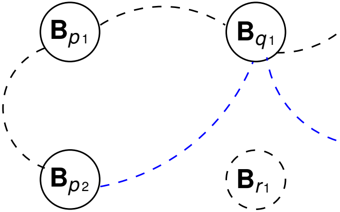
- for each $\mathbf{B}_{q_1 r_1} \in \mathcal{L}_n$: identify $\mathbf{B}_{q_1}$ and $\mathbf{B}_{r_1}$





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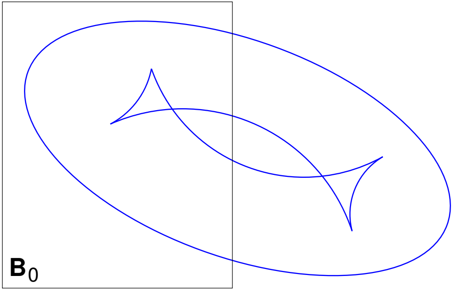

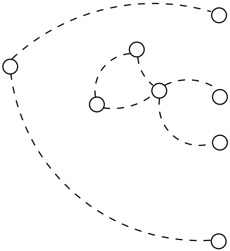

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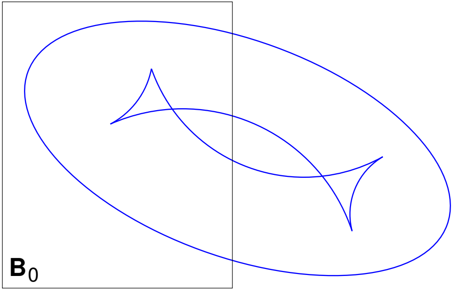
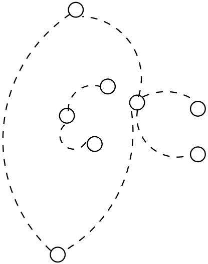
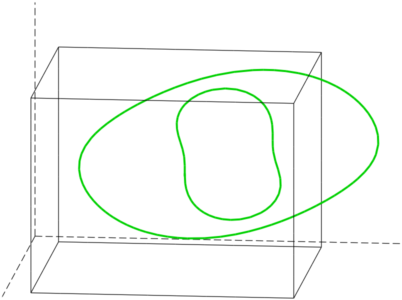




Certified numerical tools: 1-dim solver

$F : \mathbb{R}^3 \to \mathbb{R}^2$, $\mathbf{C}_0$ a box of $\mathbb{R}^3$
$\mathcal{C} = \{C \in \mathbf{C}_0 | F(C) = 0\}$ is a smooth curve of $\mathbb{R}^3$
$\mathcal{C}^1, \ldots, \mathcal{C}^m$: connected components of $\mathcal{C}$



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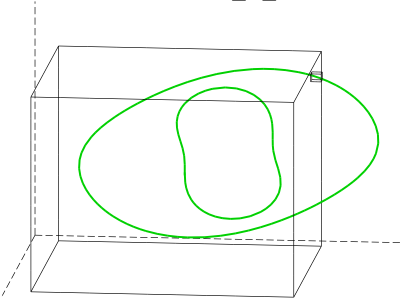
$\mathcal{C}^1, \ldots, \mathcal{C}^m$: connected components of $\mathcal{C}$

Certified path-tracker:

Input: $F : \mathbb{R}^3 \rightarrow \mathbb{R}^2$, $\mathbf{C}_0$ box of $\mathbb{R}^3$, $\epsilon \in \mathbb{R}_*^+$
An initial box $\mathbf{C} \in \mathcal{C}^i$

Output: a sequence of boxes $\{\mathbf{C}_k\}_{1 \leq k \leq l}$ enclosing $\mathcal{C}^i$.



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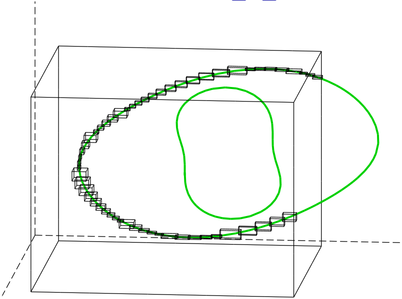
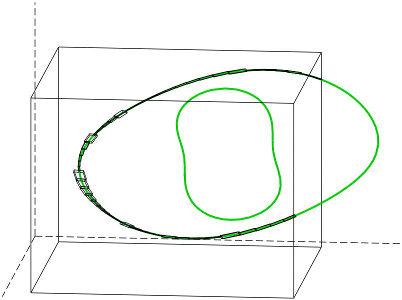


Certified numerical tools: 1-dim solver

[MGGJ13] B. Martin, A. Goldsztejn, L. Granvilliers, and C. Jermann.
Certified parallelotope continuation for one-manifolds.
SIAM Journal on Numerical Analysis, 51(6):3373–3401, 2013.

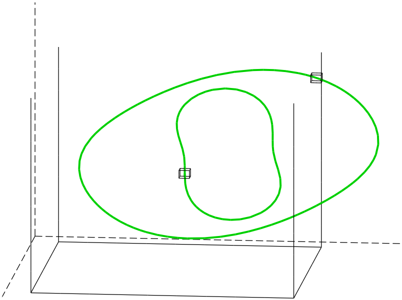


Certified numerical tools: 1-dim solver

$F : \mathbb{R}^3 \to \mathbb{R}^2$, $\mathbf{B}_0$ a box of $\mathbb{R}^2$
$\mathcal{C} = \{C \in \mathbf{B}_0 \times \mathbb{R} | F(X) = 0\}$ is a smooth curve of $\mathbb{R}^3$
$\mathcal{C}^1, \ldots, \mathcal{C}^m$: connected components of $\mathcal{C}$

Assumption (A1): $\mathcal{C}$ is compact over $\mathbf{B}_0$
(A1) holds for generic polynomials P, Q

Finding one point on each connected component

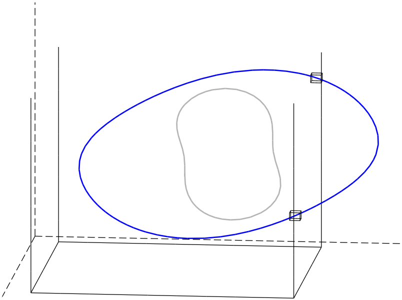


Finding one point on each connected component

Assumption (A1): $\mathcal{C}$ is compact over $\mathbf{B}_0$

Lemma: If (A1) holds, $\mathcal{C}^k$ is

- either diffeomorphic to $[0, 1]$
  $\Rightarrow$ has 2 intersections with $\partial\mathbf{B}_0 \times \mathbb{R}$
- or diffeomorphic to a circle
  $\Rightarrow$ has at least two x-critical points



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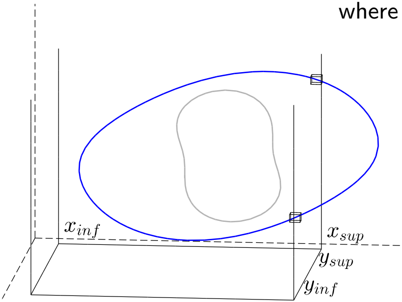

$\mathcal{C} \cap (\partial\mathbf{B}_0 \times \mathbb{R})$ are the solutions of the 4 systems:

$$\begin{cases} P(x = a, y, z) = 0 \\ Q(x = a, y, z) = 0 \end{cases}$$

$$\begin{cases} P(x, y = b, z) = 0 \\ Q(x, y = b, z) = 0 \end{cases}$$

where $a \in \{x_{inf}, x_{sup}\}$,
$b \in \{y_{inf}, y_{sup}\}$

Finding one point on each connected component

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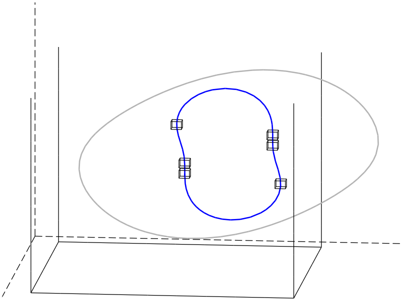

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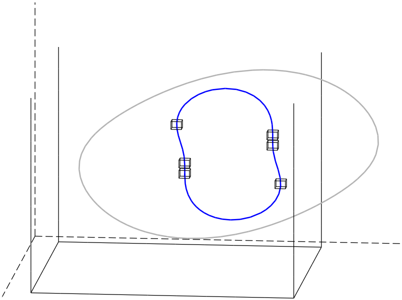
x-critical points of $\mathcal{C}$ are the solutions of the system:

$$\begin{cases} P(x,y,z) & = 0 \\ Q(x,y,z) & = 0 \\ \begin{vmatrix} P_y & P_z \\ Q_y & Q_z \end{vmatrix}(x,y,z) & = 0 \end{cases}$$



Results:

Datas: Random dense polynomials of degree d, bit-size 8

0-dim solver: multi-precision subdivision solver, `c++/cython/sage`

Path tracker: prototype in `python/cython`

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